Part II: Survey Observations and Adjustments

2.1 Introduction

While the observations made in a survey are important, it is more important to analyze the results made from that survey. The analysis can be superficial in that the surveyor looks and makes a judgement on the observations or it can be elaborate based on statistical principles from a least squares adjustment. This latter approach to analysis is the method that will be used for MDOT projects and contracts. The main advantage of least squares lies in the ability to statistically reach a conclusion as to the quality of the observations within a survey and to test to see if these measurements meet recognized standards in the profession.

2.2. Planning and Design of the Survey

One of the most important aspects of any survey project is the planning and design of that survey. With the advent of the global positioning system (GPS), planning has become more common place. The idea behind planning is to optimize the geometry of the survey thereby increasing the resulting accuracy of that survey. For example, in conventional traversing it is important to stay away from "thin" or "skinny" triangles. These kinds of geometric figures create weaknesses within the solution. Small errors in the triangles become magnified in the adjustment process. The same situation applies to GPS. Here the geometry involves the location of the satellites in the sky. As shown in **Part V (10.8)**, poor satellite geometry can result in a dilution of precision that is unacceptable for the GPS survey. Figure 2.1 shows good GPS satellite geometry whereas Figure 2.2 shows poor geometry since all of the satellites are located within one quadrant of the sky creating thin angles at the receiver station.

Figure 2.1
Example of good satellite geometry

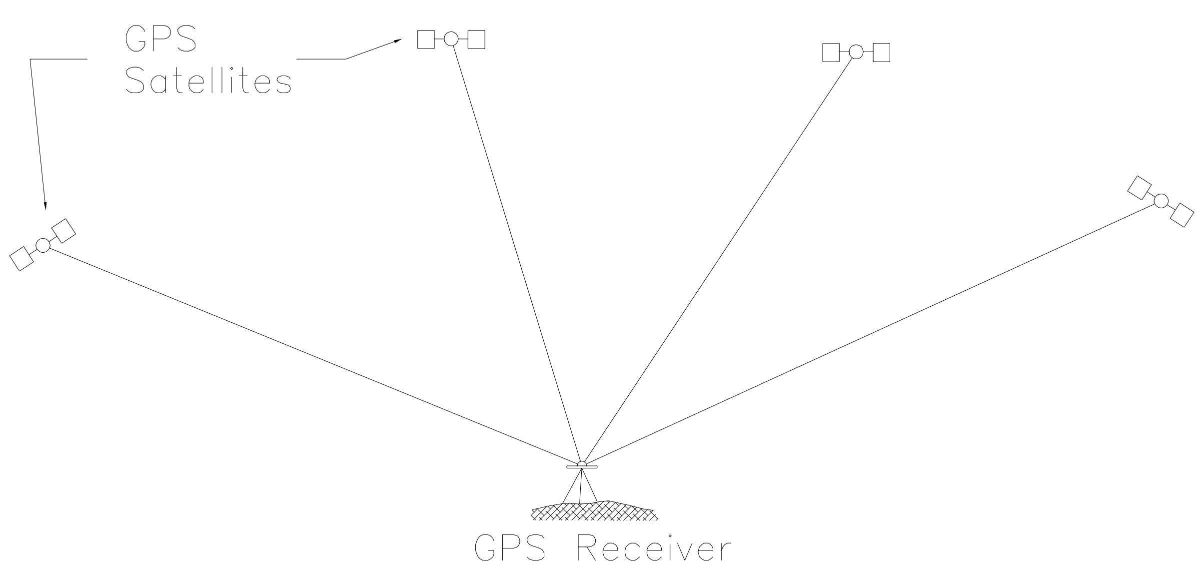
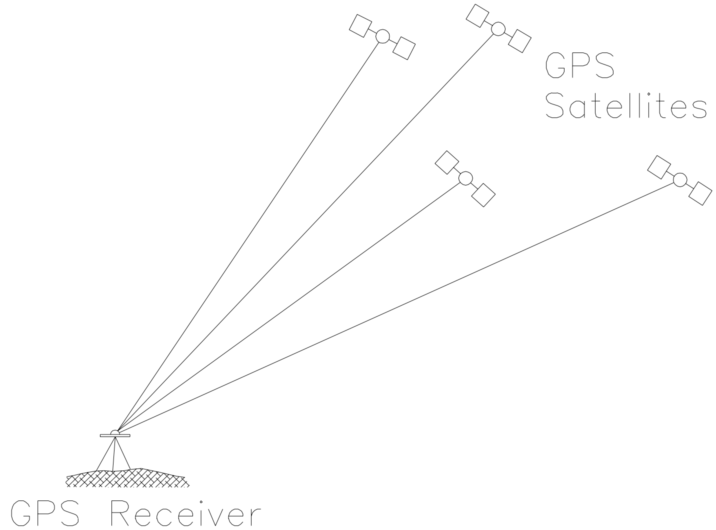


Figure 2.2
Poor satellite geometry.



Planning the survey also involves identifying the land owners upon whose land the survey will be performed. Prior to the execution of any survey, these individuals must be contacted and permission obtained to access survey points on their property. Try to identify the location of survey points to be as unobtrusive as possible and situated in locations that minimize damage to the owners property. Make sure that the owners know when the survey will be performed and the purpose for the survey.

2.3. Pre-Analysis

The ability to perform pre-analysis on a particular survey is a very helpful tool in developing the necessary specifications to support the accuracy needs of the survey. By careful planning, apriori estimates of the measured quantities can be determined and then, through error propagation, initial results based on that survey design, can be determined. The pre-analysis will help identify weaknesses in either the geometry or in the instrumentation/survey techniques that are applied to the survey.

When using mixed survey equipment for a survey, it is important that the precision of those tools be compatible to each other. For example, when using an EDM with a theodolite the relative precision of both instruments must be consistent with each other in order to achieve the desired accuracy for that particular survey. For example, Table 2.1 shows the allowable angular error for a given linear accuracy. Table 2.2 gives a comparable example identifying the allowable linear error for a given angular error.

2.4. Observations and Errors

It is a well recognized fact that all measurement, except counting, contain errors. These are inherent in any measurement because of residual errors in the manufacture of the instruments, human observations, and environmental conditions. A well designed survey will eliminate blunders and mistakes, account for any systematic errors within the measurement process, and minimize the effects of random errors that will still exist after blunders and systematic errors have been either eliminated or corrected.

Table 2.1
Allowable Angular Error for a Given Linear Error

Accuracy of Linear Measurements	Allowable Angular Error
1:500	6' 53"
1:1,000	3' 26"
1:5,000	0' 41"
1:10,000	0' 21"
1:50,000	0' 04"
1:100,000	0' 02"
1:1,000,000	0' 00.2"

2.4.1. Blunders

Blunders are mistakes. They must be eliminated from the observational set by re-observation. Blunders cannot be corrected for because they are not errors. Typical blunders include writing down the wrong number for an observation such as recording an angle of 48° 51' 25" when the instrument operator actually read 84° 51' 25". Blunders are caused by carelessness and sloppy field procedures. A common blunder found today with GPS is the erroneous recording of the height of the receiver. Blunders can be minimized by establishing field procedures in which checks

Table 2.2
Allowable Linear Error for a Given Angular Error

Least Reading of Instrument	Allowable Linear Error In			
	100'	500'	1,000'	5,000'
5'	0.145	0.727	1.454	7.272
1'	0.029	0.145	0.291	1.454
30"	0.015	0.073	0.145	0.727
20"	0.010	0.049	0.097	0.485
10"	0.005	0.024	0.049	0.242
5"	0.002	0.012	0.024	0.121
2"	0.001	0.005	0.010	0.048
1"		0.002	0.005	0.024

on the measurements are incorporated within the survey procedure. These can include very simple tasks like recording the height of the receiver in both meters and feet and checking these measurements before making the observations, or having the note keeper repeat the measurement values back to the observer before writing them in the notebook.

2.4.2. Systematic Errors

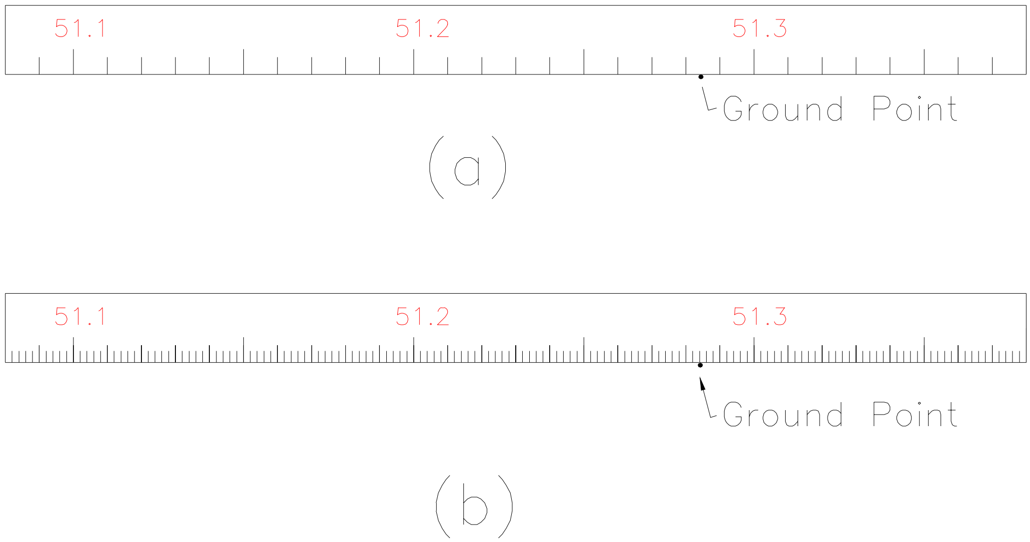
Systematic errors are those errors caused by either the instrument, operator, or environment that affect a survey measurement. They follow some known physical law and their effects can be corrected for in subsequent processing of the data. For example, in taping, systematic errors can occur in the tape itself in that the nominal length is different than the calibrated length. While a 30 meter tape may be graduated throughout, careful calibration may show that the actual tape length is 29.989 meters. Thus, every tape length results in an error of -0.011 meters. Subtracting the error from the measurement, or adding a correction, results in the correct length. The tape can also be affected by the environment. Tapes expand and contract as the temperature changes. Thus, if the tape is used when the temperature is 5°C, each tape length will again be too short. The correction can be found by taking into account the difference in the field temperature and the standard temperature and applying this to the coefficient of thermal expansion. The tape can also be affected by the way the operator uses that tape in the field measurement. If the distances were measured along the slope of the ground, then the measurements must be reduced to the corresponding horizontal length. If the tape was suspended above the points in the measurement then the correction for the effects of sag must be considered in the reductions. Finally, if the applied tension was different than the standard tension this also results in an erroneous measurement. The key thing in all of these examples is that there exists mathematical formulas in which the actual tape measurements can be corrected for in subsequent reductions.

Another principle should be evident from the example of taping and this is that systematic errors have a tendency to accumulate. All instruments and measurements have the opportunity to exhibit systematic errors. The goal of the surveyor is to ascertain if a systematic error exists, determine its magnitude, and correct the measurement accordingly.

2.4.3. Random Errors

After all of the blunders have been eliminated and the systematic errors accounted for, error still remains. This error is referred to as random error. It exists because of the limitations in the manufacture of the equipment, inconsistencies in repeated observations, and inappropriate modeling of the environmental effects on the survey process. Lets look to the taping example to see the effects of random error. In Figure 2.3(a) we see a portion of a tape graduated to 0.01 units with the mark to be measured somewhere between 51.28 and 51.29. Because it looks to be in the middle, most surveyors would record this as 51.28. In Figure 2.3(b), the same distance is now measured with a tape graduated to 0.002 units. Here the measurement would be recorded as 51.284 units. Another tape, graduated to smaller units would yield similar results. The difference of 0.004 units is a random error due to the resolution of the measuring tool.

Figure 2.3
Using two tapes to show the effects of random error.

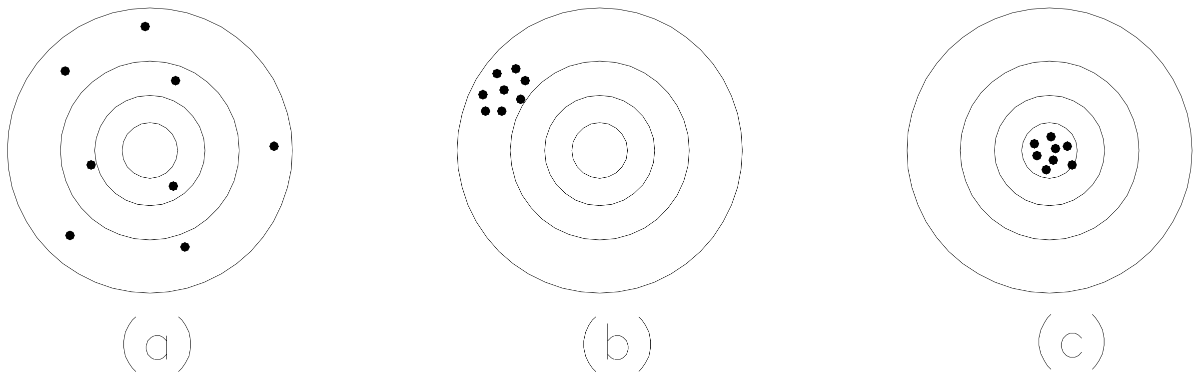


Random errors are characterized as equally likely to be positive or negative. They also have a tendency to cancel out their effects. For example, if the mark in Figure 2.3(a) was slightly to the right, then the error would have been of opposite sign. It is the random errors that are evaluated in the analysis of a survey. Random errors are unavoidable. While a careful set of measurement procedures can be implemented that minimize their effect, these errors can never be completely eliminated. The magnitude of these errors is unknown but they can be estimated through experience or subsequent analysis. Random errors follow the laws of probability and compensation in that large errors rarely occur, small errors are more likely than large errors, and there is an equal opportunity of positive or negative error to occur.

2.5. Accuracy and Precision

Two measures of quality of a series of survey measurements are accuracy and precision. Although many use these terms synonymously, they are different as shown in the targets depicted in Figure 2.4. Accuracy is how close a set of observations are to the "true" value. For example, in Figure 2.4(a), accuracy may be high when one considers all of the shots on the target. The average value may be right in the center of the target. Precision is a measure of the repeatability of a set of measurements. For example, Figure 2.4(b) shows that the difference in the 8 shots on the target is very small. In other words, there is a high level of repeatability in each of the shots that were fired. Yet, none hit the center of the target therefore while the precision was high, the accuracy was not. Finally, Figure 2.4(c) shows the same target with both high precision and high accuracy.

Figure 2.4
Targets showing the relationship between accuracy and precision.



2.6. Variance, Covariance and Correlation

Analysis of the measurements made in a survey is done by looking at redundant observations. This is very important because without these extra observations, no checks are possible within the survey. Lets look at a very simple example, a triangle. It is well known that three elements of a triangle must be known to solve for the other unknown elements. Thus, if two angles are known, the third can be found by subtracting the sum of the two known angles from 180°. But, there is no guarantee that the angle is correct. It only follows the basic rules of Euclidean geometry. On the other hand, if the third angle is also measured then there is a check on the condition that the sum of the angles must equal 180°. This is redundancy. Surveyors try to apply checks whenever possible. Angles and distances are measured multiple of times to ensure that the measurements are "good". Traverses and level loops should form closed figures, closing on the beginning point or another control/bench mark. Again, this provides redundancy and a basic check on the survey results. Redundancy is also referred to as degrees of freedom. Using the triangle example, measuring all three angles (along with one of the distances) yields one degree of freedom, or one redundant observation.

Analysis begins by looking at the errors that exist in a series of measurements. Theoretically, error is the difference between an observation and its true value. Unfortunately, the true value is not always known. Therefore, the error is estimated as the difference between the observed value and the expected value of that measurement. This is often referred to as a residual. Thus, the residual, v, can be shown to be

$$v_i = x_i - \bar{x}$$

where: x_i is the observed value, and
$\bar{x}_i$ is the expected value (such as an average).

The precision of a set of observed data can be defined by the variance. For surveying applications, it is essential to understand that the measurements represent only a sample. Thus, the sample variance gives an unbiased estimate of the population and it is defined as:

$$S^2 = \frac{\sum_{n=1}^{n} v_i^2}{n - 1}$$

From this, the standard deviation can be found as the square root of the sample variance. Thus,

$$S = \sqrt{\frac{\sum_{n=1}^{n} v_i^2}{n - 1}}$$

Lets look at an example. Assume that we have taken 25 readings from a 1" theodolite. These are given in Table 2.3. The arithmetic mean of the observations is found by summing the observations and dividing this sum by the number of observations taken. Thus,

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$$

$$= 808.\frac{5}{25} = 32.3$$

Table 2.3
Observations made with a 1" theodolite recorded to the second only.

30.1	31.2	32.9	31.0	32.7
31.4	33.4	31.7	35.4	30.4
35.9	30.9	34.0	31.0	33.1
31.9	30.5	32.3	31.8	32.6
31.7	35.3	33.0	31.1	33.2

The sample variance becomes

$$S^2 = \frac{(30.1 - 32.3)^2 + (32.7 - 32.3)^2 \quad (31.1 - 32.3)^2}{25 - 1} = 2.4917$$

Taking the square root results in a standard deviation of S = ±1.58. Note that sometimes the variance and standard deviation are designated using the Greek symbol σ. Theoretically σ represents the population standard deviation.

Correlation exists when two or more observations are jointly involved. An excellent example of correlation is the use of coordinates. Correlation is measured statistically by using the covariance. Correlation values range from -1 to +1. When the value is close to +1, then the two measurements are said to be highly correlated. On the other hand, when the correlation coefficient is close to -1 then the two variables are inversely related, such as one finds with barometric leveling where elevation is inversely proportional to atmospheric pressure. The correlation coefficient is given as

$$\hat{\rho} = \frac{s_{xy}}{s_x s_y}$$

where:

$$s_{xy} = \sum \frac{(x - \bar{x})(y - \bar{y})}{n - 1}$$

$$s_x^2 = \sum \frac{(x - \bar{x})^2}{n - 1} \qquad s_y^2 = \sum \frac{(y - \bar{y})^2}{n - 1}$$

2.7. Error Ellipses

Errors within a survey have a tendency to accumulate through the length of the survey. Lets look at a simple example of conventional traversing (figure 2.5). A target is centered over the backsight, A, while the theodolite is centered over another known point B. Centering is done using optical plummets. The unknown point, C, also has a target centered over its position and the angle is measured at point B. In this scenario, there are a number of potential sources of error affecting the measured angle. First, the optical plummets may not be perfectly adjusted resulting in the targets and instrument being displaced over the point by some small amount. Second, when the instrument operator turns the angle there are potential errors in both pointing the instrument on the target and in reading of the angle. Finally, the theodolite has certain inherent errors that will affect the measured angle. These instrument errors are described in **Appendix D**. The end result is that the "true"position of point C actually lies within a circle of uncertainty. Now, if the instrument is moved to point C and a new point D is subsequently measured, the uncertainty at D contains not only the uncertainty in the angle measurement at C but also the uncertainty in the position of point C. It is the surveyor's responsibility to measure how this error propagates throughout the survey.

A measure of this circle of uncertainty is called the error ellipse. A typical error ellipse is depicted in figure 2.6. If the standard deviations in the x and y directions are identical then the ellipse will form a circle. In addition, seldom will the u, v axes of the error ellipse follow the x and y axes of the coordinate system. The angle t is the rotation angle between the ground coordinate system and the axis of the ellipse.

Figure 2.5
Example of Error Sources in an Angle Measurement

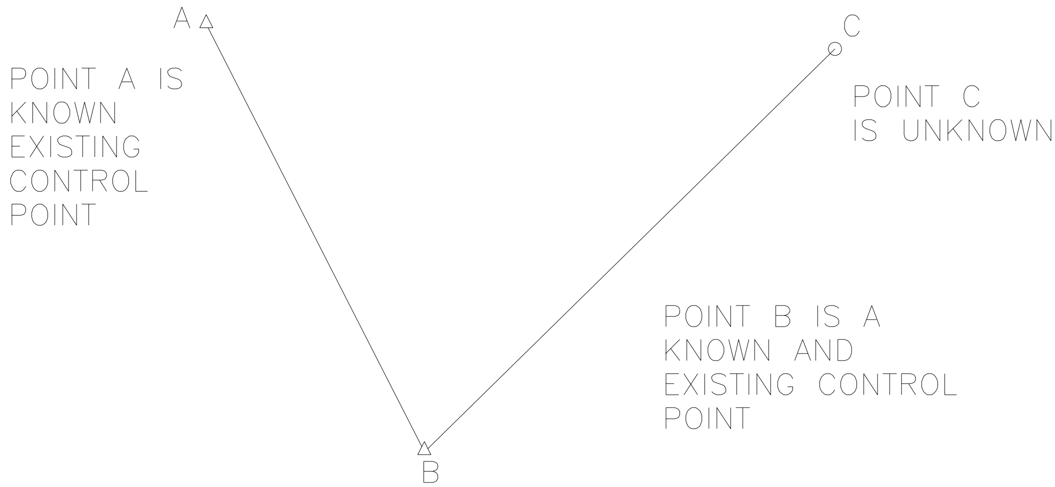
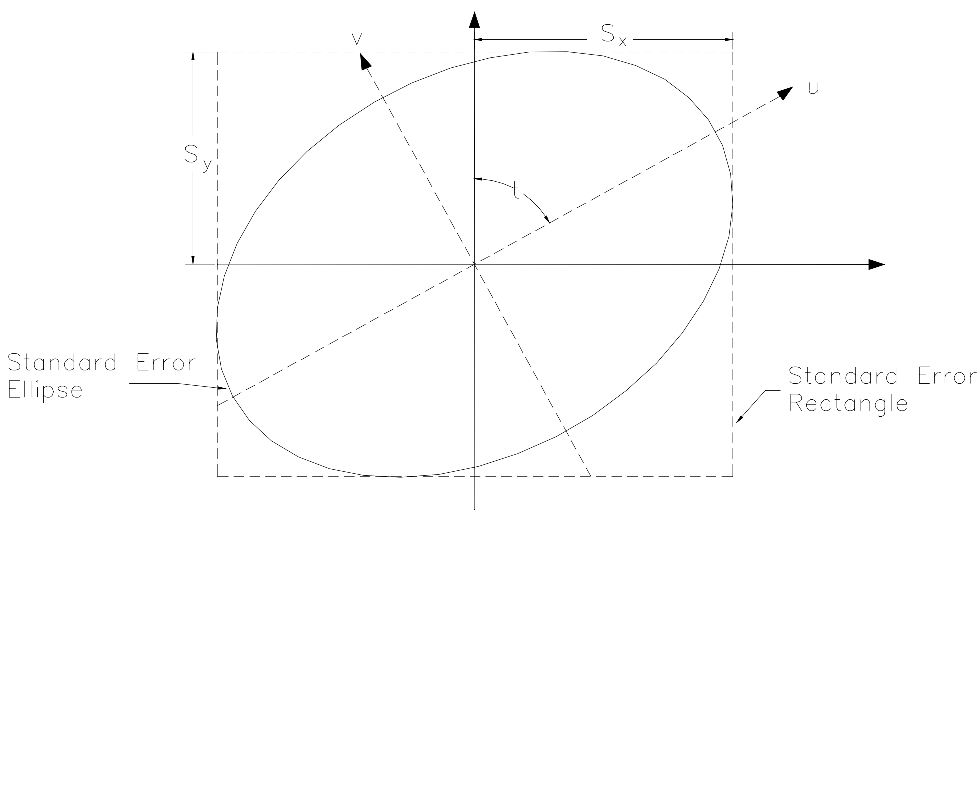


Figure 2.6
Typical error ellipse.



The semimajor and semiminor axes of the error ellipse can be computed as follows:

$$S_x^2 = \frac{S_x^2 + S_y^2}{2} + \left[\frac{\left(S_x^2 - S_y^2\right)^2}{4} + S_{xy}^2 \right]^{1/2}$$

$$S_x^2 = \frac{S_x^2 + S_y^2}{2} - \left[\frac{\left(S_x^2 - S_y^2\right)^2}{4} + S_{xy}^2 \right]^{1/2}$$

There are several advantages of the error ellipse in analysis. It provides information on the precision of the adjusted positions of the stations. In addition, the user can make visual comparisons of the relative precisions between any stations by looking at the shape of the ellipses, their sizes, and the orientation of each ellipse. These comparisons, though, need to be tempered in that the constraints used in the adjustment, the precision of the observations, and the geometry of the survey will all affect the error ellipse. Moreover, it is general practice to enlarge the ellipse for visual purposes. Finally, the error ellipses can be used in pre-analysis to help select the equipment and the field techniques that will be used to complete the survey.

2.8. Error Propagation

As we have seen in the previous section, error propagates through a survey affecting the adjustment of subsequent points within the survey. Error propagation does not necessarily imply that a least squares solution to a survey problem is necessary, although it is a convenient byproduct of such an adjustment. To determine the effects of errors made in a survey, one must propagate that error through the survey.

Assume that we have a function Z. Then, the law of propagation of variances can be shown to follow this simple rule:

$$S_Z = \sqrt{\left(\frac{\partial Z}{\partial X_1} S_{x_1}\right)^2 + \left(\frac{\partial Z}{\partial X_2} S_{x_2}\right)^2 + \cdots + \left(\frac{\partial Z}{\partial X_p} S_{x_p}\right)^2}$$

The basic assumption in this formula is that no correlation exists between the different measurements $S_{x_i y_i} = 0$. Using this basic formulation, some frequently used functions can be defined. The standard deviation of the sum can be found to be

$$S_{sum} = \sqrt{S_1^2 + S_2^2 + \cdots + S_n^2}$$

The standard deviation for a series is

$$S_{series} = S_x \sqrt{n}$$

Finally, the standard deviation of the mean can be shown to be

$$S_{mean} = \sqrt{\left(\frac{1}{n}S_{\bar{x}_1}\right)^2 + \left(\frac{1}{n}S_{\bar{x}_2}\right)^2 + \cdots + \left(\frac{1}{n}S_{\bar{x}_n}\right)^2} = \sqrt{\frac{nS^2}{n^2}} = \frac{S}{\sqrt{n}}$$

In a more general sense, one can define the variance-covariance matrix for a function Z of n independently measured quantities as $\sum_{ZZ}$. This will be defined as:

$$\sum_{ZZ} = \begin{bmatrix} \frac{\partial Z_1}{\partial x_1} & \frac{\partial Z_1}{\partial x_2} & \cdots & \frac{\partial Z_1}{\partial x_n} \\ \frac{\partial Z_2}{\partial x_1} & \frac{\partial Z_2}{\partial x_2} & \cdots & \frac{\partial Z_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \cdots \\ \frac{\partial Z_n}{\partial x_1} & \frac{\partial Z_n}{\partial x_2} & \cdots & \frac{\partial Z_n}{\partial x_n} \end{bmatrix} \begin{bmatrix} S_{x_1^2} & S_{x_1x_2} & \cdots & S_{x_1x_n} \\ S_{x_1x_2} & S_{x_2^2} & \cdots & S_{x_2x_n} \\ \vdots & \vdots & \ddots & \vdots \\ S_{x_1x_n} & S_{x_1x_n} & \cdots & S_{x_n^2} \end{bmatrix} \begin{bmatrix} \frac{\partial Z_1}{\partial x_1} & \frac{\partial Z_2}{\partial x_1} & \cdots & \frac{\partial Z_n}{\partial x_1} \\ \frac{\partial Z_1}{\partial x_2} & \frac{\partial Z_2}{\partial x_2} & \cdots & \frac{\partial Z_n}{\partial x_2} \\ \vdots & \vdots & \ddots & \cdots \\ \frac{\partial Z_1}{\partial x_n} & \frac{\partial Z_2}{\partial x_n} & \cdots & \frac{\partial Z_n}{\partial x_n} \end{bmatrix}$$

In matrix notation, this is

$$\sum_{ZZ} = A \sum A^T$$

where $\sum$ is the variance-covariance matrix of the observations. This form does show the general law of propagation of variances where correlation in the observations may exist.

2.9. Quality Checks/Quality Assurances

After the least squares adjustment has been performed, the next step is to evaluate the results statistically. Since most survey measurements represent only a small sample of the total population, analysis of a normal distribution is not the most appropriate measure of the quality of the performance. Instead, two main statistical distributions are used: Chi-square (χ^2) and the t or student-t distributions.

2.9.1. Chi Square (χ^2)

The χ^2 distribution is used to test the sample variance to see if it is in agreement with the population variance. A confidence region is established in which the sample variance should lie between. The actual location of the sample variance is based on some specific percentage probability, and the sample set variance, and the degrees of freedom for the particular problem. It is defined by

$$\chi^2 = \frac{(n - 1)S^2}{\sigma^2}$$

To test the variance, a confidence region is created.

$$\frac{dfS^2}{\chi^2_{\alpha/2}} < \sigma^2 < \frac{dfS^2}{\chi^2_{1-\alpha/2}}$$

where df is the degrees of freedom and (1 - α) is the confidence interval. Lets look at an example. An angle was measured 10 times and found to be 42° 12' 14.6" with a standard deviation, s, of ±3.7". Perform a chi-square test to see if the standard deviation is 2.0" at the 5% confidence level versus the alternative that it is not 2.0".

Modify the previous equation and look in the χ^2 table for the appropriate χ^2 values at the 0.975 and 0.025 confidence levels with 9 degrees of freedom. Thus,

$$\frac{\chi^2_{0.975}\sigma^2}{n - 1} = \frac{(2.80)(2.0)^2}{9} = 1.20''$$

$$\frac{\chi^2_{0.025}\sigma^2}{n - 1} = \frac{(19.0)(2.0)^2}{9} = 8.44$$

Since the sample falls within the range, the hypothesis is accepted.

2.9.2. Student-t

The second important statistical test is the t- or student-t distribution which is used to compare the sample mean to the population mean. This distribution is similar, but more variable or flatter, to the normal distribution. The t-test is used in surveying because of the small sample of observations that are made within our surveys. The distribution is shown as:

$$t = \frac{\hat{x} - \mu}{S/\sqrt{n}}$$

As an example, refer to the theodolite observations that were presented in Table 2.3. Recall that the arithmetic mean was 32.3" and the standard deviation was computed to be ±1.58". A total of 25 pointings of the theodolite were made. Then, lets compare this standard error to the population mean at the 95% confidence level.

We begin by setting up the confidence interval for the t-distribution as

$$\bar{x} - t_{\left(\frac{\alpha}{2}\right)} \frac{S}{\sqrt{n}} < \mu < \bar{x} + t_{\left(\frac{\alpha}{2}\right)} \frac{S}{\sqrt{n}}$$

where: μ is the arithmetic mean for the population and
α is the percentage for the confidence level desired.

Because the t-distribution shows the area under the distribution curve at the ends, the confidence level is (1 - α) where α in this case is 0.05. Since 25 observations were made, there are 24 redundant observations which means that the degrees of freedom (df) is 24. Using a table for the t-distribution, $t_{\left(\frac{\alpha}{2}\right)} = t_{\left(\frac{0.05}{2}\right)} = 2.064$. Then

$$32.3 - 2.064 \left(\frac{1.58}{\sqrt{24}} \right) < \mu < 32.3 + 2.064 \left(\frac{1.58}{\sqrt{24}} \right)$$

which results in

$$31.6'' < \mu < 33.0''$$

This means that the population average will fall, at the 95% confidence level, between 31.6" and 33.0".

If we took just the first column of the data in Table 2.3, we would have the following:

$\bar{x} = 32.2''$
S = ± 2.18"
df = 4

Then,

$$32.2 - 2.776 \left(\frac{2.18}{\sqrt{4}} \right) < \mu < 32.2 + 2.776 \left(\frac{2.18}{\sqrt{4}} \right)$$

yielding

$$29.2'' < \mu < 35.2''$$

This shows that, with increasing observations, the student-t distribution gets closer to the normal distribution.