A COMPUTER PROGRAM FOR COMPUTING PROBABILITIES AND GENERATING RANDOM VARIATES FOR THE GAMMA, NORMAL, AND CHI-SQUARE DISTRIBUTIONS AND ITS APPLICATIONS TO HIGHWAY TECHNOLOGY

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Introduction

Very often highway research and testing require the simultaneous measurement of interrelated variables. For example, the gradation test of 22A aggregate measures the percentages passing the 1-in., 3/4-in., 3/8-in., and No. 8 sieves and the percentage loss-by-washing. How would one conduct experiments to collect this kind of data for the purpose of designing an in-place acceptance sampling plan? Of course, the complete literature on univariate acceptance sampling plans is readily available; unfortunately, however, this literature provides very little information about designing multivariate acceptance sampling plans. This could be because the statistical theory is quite complicated in the multivariate case. In addition to the theoretical difficulties, considerable practical difficulty could be encountered in computing the probabilities pertaining to complex distributions such as the multinormal. Although many different numerical integration methods can be used to compute single integrals; they are time-consuming in the multivariate problems often encountered in highway technology. In these cases, the methods of simulation would be of considerable value. In order to compute probabilities involving, for example, the multinormal distribution by simulation, we need a 'fast' computer program for generating normal variates.

It is well known that the normal distribution can be obtained, through a defined equation, from the gamma distribution. Moreover, the chi-square and Erlang distributions are special cases of the gamma distribution. In the field of traffic engineering, many variables appear to be distributed according to the Erlang distribution. For example, the distribution of gaps in a major traffic stream intersected by either a minor street or an entrance ramp is known to be Erlang. The angle of vehicle encroachments off roadways and on to shoulders was found to be gamma distributed. The distribution of the time between two vehicles entering a roadside inspection station, for safety or emission control purposes, could be Erlang if the headway between two vehicles on the roadway is exponentially distributed and a special sampling procedure is used to select vehicles for inspection. To design suitable ramp metering systems and inspection programs, many criteria have to be checked using the available data. Quite often this task is very difficult because the explicit solution is not known. In this circumstance simulation is usually used to obtain the approximate solution. This means that a 'fast' computer program for generating gamma variates is needed, and could be very useful in highway research.

The major purpose of this report is to present an algorithm for writing such a computer program. Six practical examples in highway research and testing are presented to demonstrate the need for this computer program.

Listings of a FORTRAN computer program based on techniques presented in this report together with users' instructions are included in the Appendix.

Example 1 - Aggregate Gradation Distribution

Let X_i, $i = 1, 2, 3$, and 4, be the percent passing the 3/4-in., 3/8-in., and No. 8 sieves, and the percent loss-by-washing, respectively. For the purpose of designing a meaningful acceptance sampling plan, it is necessary to know the aggregate quality produced by the manufacturing process. Specifically, if a random sample is selected from a truckload of aggregate produced by a manufacturing process set at the target value μ_i, $i = 1, 2, 3$, and 4, with the variation described by the covariance matrix Σ, what is the probability that the aggregate gradation of this sample will fall outside the specification limits ? The above question can be expressed in the following equation:

$$P = 1 - P_r(L_i \leq X_i \leq U_i,\ i = 1,2,3, \text{ and } 4)$$

$$= 1 - \int_{L_1}^{U_1}\int_{L_2}^{U_2}\int_{L_3}^{U_3}\int_{L_4}^{U_4} dF(x_1, x_2, x_3, x_4) \qquad (1)$$

where F is the joint distribution of X_1, X_2, X_3, and X_4, L_i and U_i are the respective lower and upper specification limits, $i = 1, 2, 3$, and 4, and P is the product quality.

In order to compute P defined in Eq. (1), the first step is to specify the joint distribution F. For this purpose, 369 data points of (X_1, X_2, X_3, X_4) were collected from an aggregate pit. These data are then analyzed as follows:

a) For demonstration purposes we transform X_i to Y_i, $i = 1, 2$, and 3, defined as the percentage retained on the sieve corresponding to the index i. The empirical frequency distribution, F_i, of each Y_i, $i = 1, 2$, and 3, is then obtained.

b) We plot the empirical distribution $\hat{F}_1(y)$ of Y_1 in Figure 1. It seems that $\hat{F}_1(y)$ is a normal distribution. To verify this, we use the Lilliefors testing procedure (Normality Test) to test the null hypothesis that the 369 observations of Y_1 were obtained from a normal population with unspecified mean and variance. The Lilliefors[1] test statistic, T_2, is defined as:

$$T_2 = \max_i \left| \hat{F}_1(y_{1i}) - N(y_{1i};\ \bar{y}, S^2) \right| \qquad (2)$$

[1] Conover, W. J., Practical Nonparametric Statistics, pp 302-306.

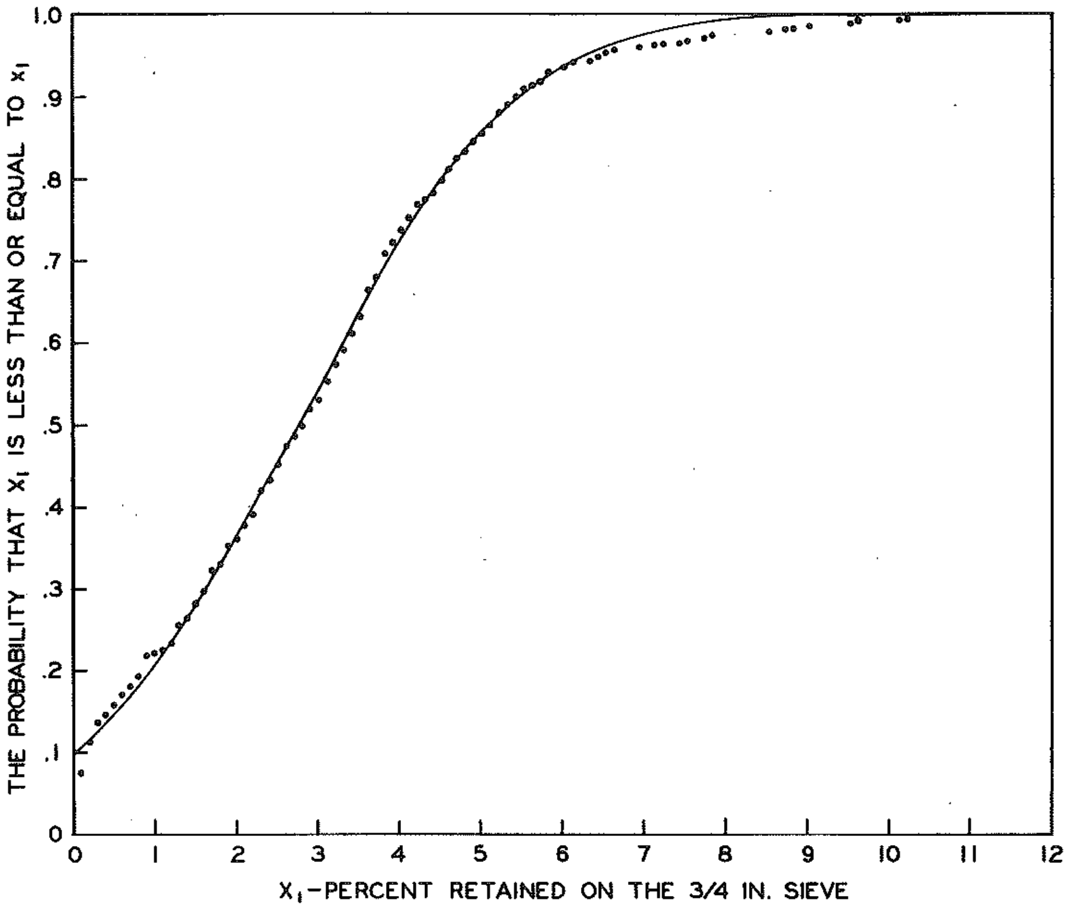

Figure 1. The empirical and fitted distribution of the aggregate percentage retained on the 3/4-in. sieve.

where y_{1i} is the i^{th} largest observation of Y_i, $F_1(y_{1i})$ is the observed frequency distribution at y_{1i}, $\bar{y}$ and S^2 are the sample mean and variance, respectively, and $N(y_{1i}; \bar{y}, S^2)$ is the value of the normal distribution (with mean $\bar{y}$ and variance S^2) at y_{1i}.

With the help of a computer program that computes the standard normal distribution, we find that $T_2 = 0.0547$. This value causes us to reject the above null hypothesis. This conclusion makes sense considering the skewness of the data.

We note that T_2 can be manually computed using the standard normal table. However, the computation would be very time-consuming if the number of distinguishing points is large.

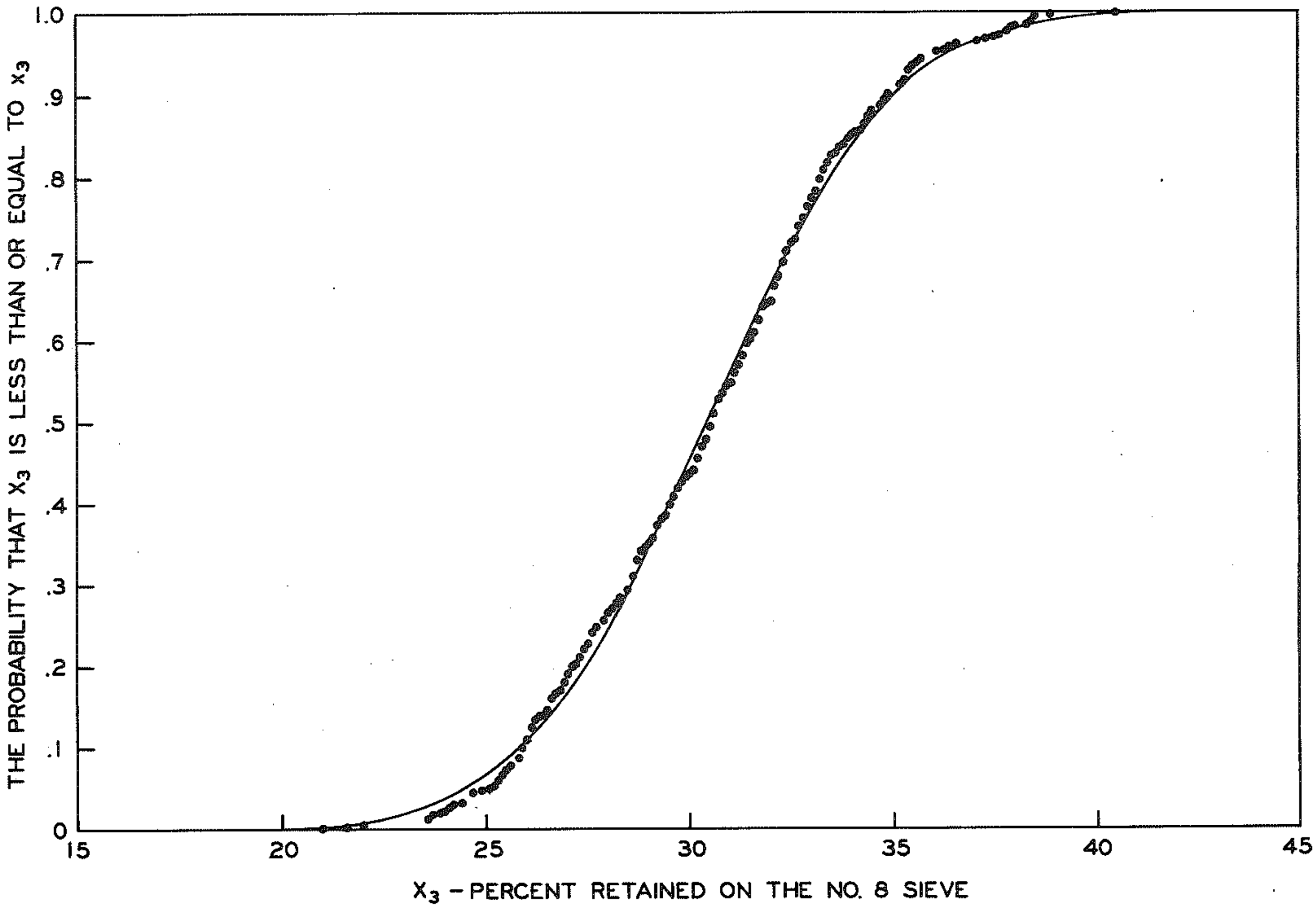

Figure 2. The empirical fitted distribution of the aggregate percentage retained on the No. 8 sieves.

c) Due to the skewness, we define F1 as follows:

$$F_1(y) = \begin{cases} 0 & \text{if } y \leq 0 \\ N(y;\mu_1,\sigma_1^2) & \text{if } 0 < y \leq 100 \\ 1 & \text{if } y > 100 \end{cases} \quad (3)$$

The above distribution is then used to fit the empirical distribution, that is, we would like to estimate μ_1 and σ_1^2 in the sense that $SSR = \sum_i \left[\hat{F}_1(y_{1i}) - F_1(y_{1i})\right]^2$ is minimal. With the help of a non-linear curve fitting computer program and a standard normal distribution subroutine, we find that $\mu_i = 2.72959$ and $\sigma_i^2 = 4.5569$. The fitted result is also plotted in Figure 1.

We note that it would not be feasible to estimate μ_i and σ_i^2 by the least squares criterion without a standard normal distribution computer program.

d) Now, we would like to test the null hypothesis that the 369 observations of Y1 were obtained from a population that has a distribution F_1, defined in Eq. (3), with parameters $\mu_i = 2.72959$ and $\sigma_i^2 = 4.5569$. We use the Kolmogorov goodness of fit test[2] to test this null hypothesis. The test statistic D_n is defined as,

$$D_n = \max_i \left| \hat{F}_1(y_{1i}) - F(y_{1i}) \right| \quad (4)$$

Again, with the help of a standard normal distribution computer program we obtain $D_n = 0.033005$ which strongly suggests that 369 data points were sampled from a population distributed according to F_1 defined in Eq. (3) with parameters $\mu_i = 2.72959$ and $\sigma_1^2 = 4.5569$.

e) Repeating the above steps on Y_2, Y_3, and Y_4, we conclude that the distribution function of Y_i, i = 2, 3, and 4, assumes the same form as Y_1 does. Moreover, $F_i(0)$ for i = 2, 3, and 4 is so small that $F_i(y)$ can be treated as $N(y;\mu_i,\sigma_i^2)$. We shall only present the empirical and fitted distributions of Y_3 in Figure 2.

f) Repeating the above procedures on the data set obtained from the original data set by deleting those observations of (Y_1, Y_2, Y_3, Y_4) such that $Y_1 = 0$, we conclude that $F_i(y)$, i = 1, 2, 3, and 4, can be well approximated by a normal distribution with some parameters μ_1 and σ_i^2.

[2] Conover, W. J., Practical Nonparametric Statistics, pp 295-298.

g) The above procedures are also used to show that distribution functions of $Y_1 + Y_2$, $Y_1 + Y_3$, and $Y_2 + Y_3$, etc., are also normal.

In order to show that Y_1, Y_2, Y_3, and Y_4 have a multinormal distribution, one would have to show that every non-trivial combination of Y_1, Y_2, Y_3, and Y_4 has a normal distribution. Thus, it is not feasible to prove rigorously that Y_1, Y_2, Y_3, and Y_4 have a multinormal distribution. However, the above analyses strongly suggest that Y_1, Y_2, Y_3, and Y_4 conditioned on $Y_1 > 0$, and Y_2, Y_3, and Y_4 conditioned on $Y_1 = 0$, have multinormal distributions. This statement can also be checked by using the chi-square test procedures. We end this section by noting that it is not feasible to perform the above analyses without a standard normal distribution computer program.

Example 2 - Aggregate Sample Size Selection

We are interested in knowing how many scoops should be taken from various locations of a truckload of aggregate to form a representative composite sample. To answer this question, we present the following conservative method.

Let X_{ij}, $j = 1, 2, \ldots, k$, be the measurement of the j^{th} component of the i^{th} random sample, where $i = 1, \ldots, M$. In Example 1, the j^{th} component will be the sieve size corresponding to index j, the i^{th} random sample could be the i^{th} scoop from a truckload of aggregate, the M will be the minimal number of scoops required to form a representative sample. Let μ_j be the mean of X_{ij} and Σ be the covariance matrix of X_{ij}, $j = 1, \ldots, k$. Denote

$$\bar{X}_j = \frac{1}{M} \sum_{i=1}^{M} X_{ij} \tag{5}$$

$\bar{X}_j$ is the sample mean of the j^{th} component, which is an unbiased estimate of the population mean μ_j. We wish to have

$$P_r\left(\left| \bar{X}_j - \mu_j \right| \leq d_j,\ j = 1, \ldots, k \right) = 1 - \alpha \tag{6}$$

where d_j is the chosen margin of error for the measurement of the j^{th} component and α is a small probability (risk). Equation (6) can be rewritten as:

$$P_r\left(\frac{\sqrt{M} \left| \bar{X}_j - \mu_j \right|}{\sqrt{\sigma_{jj}}} \leq \frac{\sqrt{M}\, d_j}{\sqrt{\sigma_{jj}}},\ j = 1, \ldots, k \right) = 1 - \alpha \tag{7}$$

where σ_{ij} is the $(ij)^{th}$ element of the matrix Σ.

Using simultaneous confidence interval techniques the required sample size M is found to be:[3,4]

$$M = \underset{j=1,\ldots,k}{\text{Max}} \left[\frac{\sigma_{jj} \chi^2_{\alpha;k}}{d_j^2} \right] \tag{8}$$

where $\chi^2_{\alpha;k}$ is the 100 α upper percentage point of the chi-square distribution with k degrees of freedom.

We note that Eq. (8) was obtained under the assumption that X_{ij}, j = 1, . . . , k, has a multinormal distribution. If the normality assumption is violated and M determined by Eq. (8) is large enough to ensure the normality of $\bar{X}_j$, j = 1, . . . , k, according to the Central Limit Theorem, M would be the required sample size. Otherwise, we should use the sample size required by the Central Limit Theorem or determined by other methods.

The sample size M determined by Eq. (8) is a conservative number in the sense that

$$P_r \left(\left| \bar{X}_j - \mu_j \right| \leq d_j, j=1,\ldots,k \right) \geq 1-\alpha \tag{9}$$

In case the sample size determined by Eq. (8) is too large for practical consideration such as manpower and testing cost, etc., the sample size can be chosen as the smallest number satisfying Eq. (9). To do this, a computer program calculating multiple integrals with a multinormal distribution function integrand is needed. That is, we need a computer program to compute,

$$\begin{aligned} P &= P_r \left(\left| \bar{X}_j - \mu_j \right| \leq d_j, j=1,\ldots,k \right) \\ &= \int_{\mu_1 - d_1}^{\mu_1 + d_1} \cdots\cdots \int_{\mu_k - d_k}^{\mu_k + d_k} dF(x_1,\ldots,x_k; \mu_1,\ldots,\mu_k, \Sigma/M) \end{aligned} \tag{10}$$

where F is the multinormal distribution with means μ_j, j = 1, . . . , k, and covariance matrix Σ/M.

As mentioned before, numerical integration methods used for computing single integrations would be very time-consuming for the case of multi-

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[3] Morrison, D. F., Multivariate Statistical Methods, McGraw-Hill Book Company, 1967.

[4] Anderson, T. W., An Introduction to Multivariate Statistical Analysis, John Wiley and Sons, Inc., 1958.

ple integrals such as those defined in Eq. (10). An alternative approach is the simulation method described below.

a) Generating a uniform random number P_1 in $(0, 1)$, we obtain x_1 such that $P_1 = F_1(x_1; \mu_1, \sigma_{11}/M)$, where F_1 is a normal distribution with mean μ_1 and variance σ_{11}/M.

b) Given $X_1 = x_1$, $(X_2, \ldots, X_k)$ has a multinormal distribution with mean μ_j', $j = 2, \ldots, k$ and covariance matrix Σ'/M, where

$$\begin{pmatrix} \mu_2' \\ \vdots \\ \mu_k' \end{pmatrix} = \begin{pmatrix} \mu_2 \\ \vdots \\ \mu_k \end{pmatrix} + \Sigma_{21} \cdot (x_1 - \mu_1)/\sigma_{11} \tag{11}$$

$$\Sigma' = (\sigma_{ij}') = \Sigma_{22} - \Sigma_{21} \cdot \Sigma_{12}/\sigma_{11} \tag{12}$$

and

$$\Sigma = \begin{bmatrix} \sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} \tag{13}$$

c) Generating a uniform random number P_2 in $(0, 1)$, we obtain x_2 such that $P_2 = F_2(x_2; \mu_2', \sigma_{11}'/M)$, where F_2 is a normal distribution with mean μ_2' and variance σ_{11}'/M and σ_{11}' is the $(1, 1)^{th}$ element of the covariance matrix Σ'.

d) Repeat steps a) through c) to obtain x_j, $j = 3, \ldots, k$.

e) Check to see whether $\mu_j - d_j \leq x_j \leq \mu_j + d_j$ for every $j = 1, \ldots, k$. If yes, add 1 to W which is defined as the number of simulation points such that $|\bar{x}_j - \mu_j| \leq d_j$, $j = 1, \ldots, k$.

f) Repeat steps a) through e) L times. L is the number of simulation points.

g) The P defined in Eq. (10) is then equal to W/L.

By repeating the above procedures for various values of d_i, $i = 1, \ldots, k$ and M, one can obtain a family of curves as shown in Figure 3. In Figure 3, a is the fraction of the predetermined numbers e_i, $i = 1, \ldots, k$. Figure 3 can be used to determine various sample sizes and their respective error margins all giving the same risk probability. From this set of plans, one can choose in practical terms the most suitable plan for the experiment.

We note again that the above task can only be accomplished with the help of a computer program that would find the solution of the equation $P = F(x; \mu, \sigma^2)$ where P is the given probability and F is the normal distribution with parameters μ and σ^2.

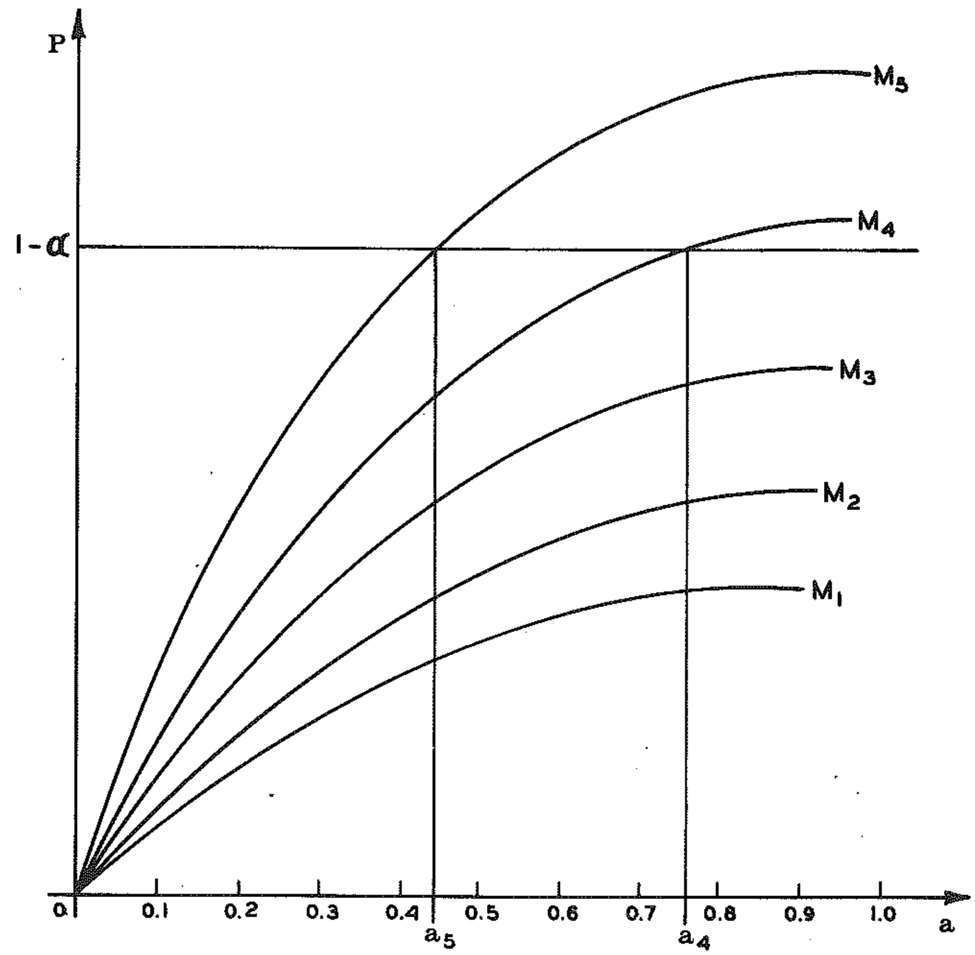

Figure 3. The relationships among the confidence coefficient (α), the margin of error and the sample size. This is a hypothetical graph for the demonstration purpose.

$$P = P_r\left(\frac{\sqrt{M}\,|\bar{x}_j - \mu_j|}{\sqrt{\sigma_{jj}}} \leqslant \frac{a\,e_j\sqrt{M}}{\sqrt{\sigma_{jj}}},\ j = 1, \ldots, k\right)$$

Example 3 - Aggregate Product Quality Determination

The ideal distribution of in-place aggregate is such that the aggregate gradation of every spot meets the specification. To ensure that the accepted project has a high degree of uniformity, we should adopt an inspection plan of so-called "acceptance sampling by attributing," such as the single sampling fraction defective sampling plan or Wald's truncated sequential probability ratio plan. These plans require that we specify the producer's risk (α), the consumers risk (β), the acceptable product quality level (P_α) and the rejected product quality level (P_β).

By using the simulation method described in Example 2, we can compute the product quality P defined in Eq. (1) corresponding to various targeted aggregate gradations. The results for a chosen Σ are presented in Table 1. By knowing the acceptable aggregate gradation, one can choose the proper P from Table 1.

TABLE 1
RELATIONSHIP OF AGGREGATE GRADATION
AND PRODUCT QUALITY

The Targeted Aggregate Gradation(μ_1, μ_2, μ_3, μ_4)				Product Quality, P
Percent Passing Sieve			Loss-By-Washing	
3/4-in.	3/8-in.	No. 8		
100.00	85.0	50.0	8.0	0.671
98.25	82.5	47.5	7.5	0.402
97.50	80.0	45.0	7.0	0.149
96.25	77.5	42.5	6.5	0.043
95.00	75.0	40.0	6.0	0.018
93.75	72.5	37.5	5.5	0.064
92.50	70.0	35.0	5.0	0.224
91.25	67.5	32.5	4.5	0.503
90.00	65.0	30.0	4.0	0.810

Again this table can be constructed only with the help of a computer program which solves the equation $P = F(x; \mu, \sigma^2)$ where P is the given probability and F is a normal distribution with parameters μ and σ^2.

Example 4 - Median Barrier Collision Probability

The question of median barrier installation at a particular location is complicated by the considerable doubts expressed in the literature as to net safety benefits. In general, it is acknowledged that any barrier, sufficiently strong to contain high velocity impact, is itself a hazard. Therefore, engineers are cautioned as to the complex, 'trade-off' nature of decision making in this area.

In order to perform cost and benefit analyses on median barrier installation, one would have to compute the following basic probabilities:

a) The probability P_B that a vehicle encroaching onto the median would collide with the median barrier if the barrier is installed s feet away from the edge.

b) The probability P_H that a vehicle encroaching onto the median would collide with vehicles on the opposite roadway if there is no barrier between the roadways.

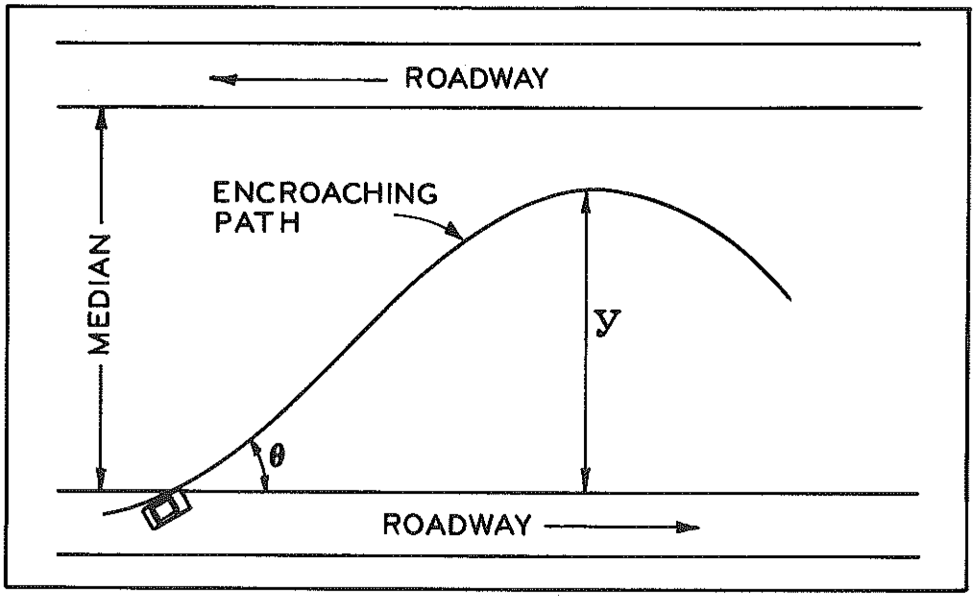

Figure 4. The angle of the encroachment and the corresponding maximum lateral distance reached by the vehicle.

To compute the probability specified in a), we need to know the distribution, $G(\theta)$, of the encroachment angle θ and the conditional distribution, $F_\theta(y)$, of the maximum lateral encroachment distance, y, given that the encroachment angle is θ. The graphical explanation of θ and y is presented in Figure 4. If these two distributions are known, the probability P_B defined in a) is,

$$P_B = \int_{\theta=0}^{\frac{\Pi}{2}} \int_{y=s}^{\infty} dF_\theta(y)\, dG(\theta) \tag{14}$$

[5] Hutchinson, J. S., and Kennedy, T. W., "Median of Divided Highways-Frequency and Nature of Vehicle Encroachments," University of Illinois, Engineering Experiment Station Bulletin 487, 1966.

Based on the Hutchinson and Kennedy data[5], we found that $G(\theta)$ is a gamma distribution with parameters $\alpha = 1.63083$ and $\beta = 5.63424$ and $F_\theta(y)$ agrees with the normal distribution for $y > 0$. The empirical and fitted $G(\theta)$ are presented in Figure 5. The probability P_H (defined in b) takes a more complicated form than Eq. (14). However, both $G(\theta)$ and $F_\theta(y)$ are also needed in computing P_H.

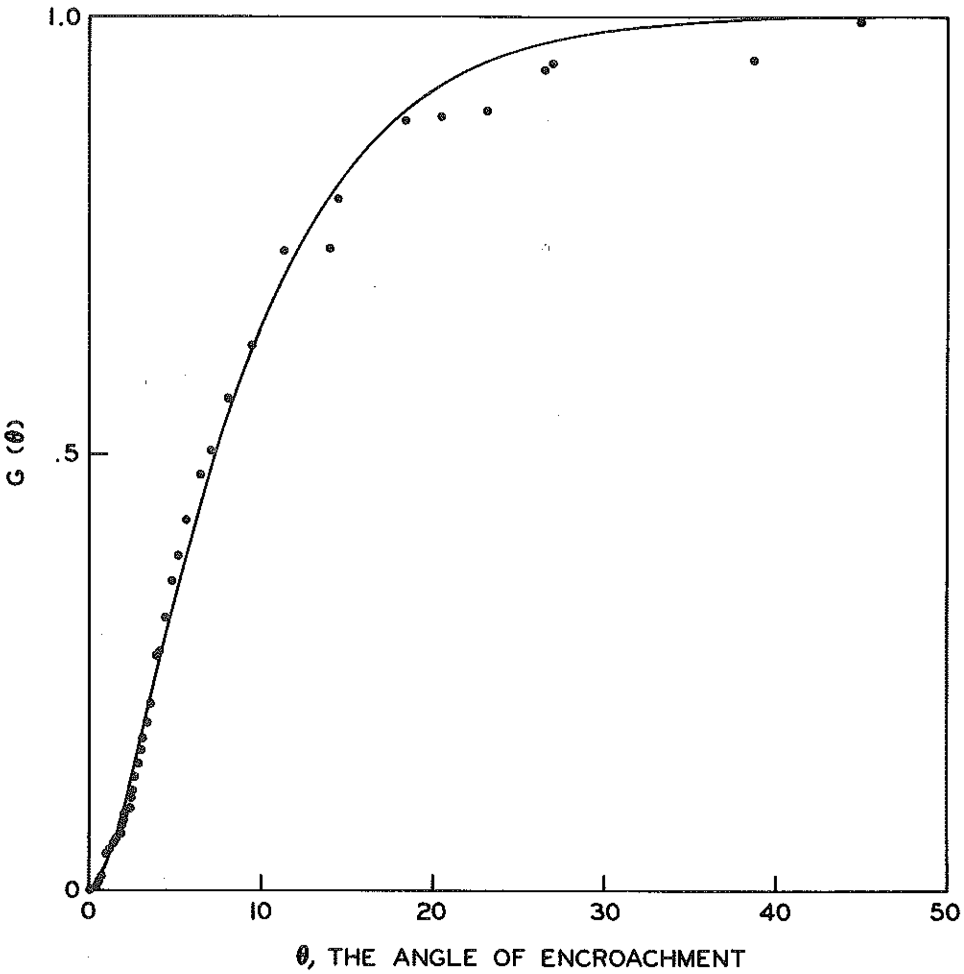

Figure 5. The empirical and fitted distributions of the roadside encroachment angles, $G(\theta)$.

As one can see from the discussion, a computer program that computes normal and gamma probabilities is needed to obtain the best fit of $G(\theta)$ and $F_\theta(y)$. After we obtain $G(\theta)$ and $F_\theta(y)$, we must compute P_B and P_H. Again, simulation probably is one of the best ways to compute Eq. (14). A computer program that generates normal and gamma variates is required for this purpose.

Example 5 - Entrance Ramp Merging Operation

Suppose that the distribution of headways between two vehicles entering an entrance ramp is described by f_q, where q is the arrival rate. These vehicles are obliged to yield to the freeway traffic, forming a single line and waiting for successive vehicles at the head of the line to merge. It may be assumed that a ramp vehicle waiting to merge assesses each time gap t in the traffic on the outside lane of the freeway until it finds an acceptable gap T. This gap length is assumed to be of sufficient length to allow safe entrance onto the freeway. The time delay of this model has been investigated by many researchers. We denote h_Q to be the distribution of time delay encountered by a ramp vehicle in merging position with Q denoting the expected time delay. It is apparent that the entrance-ramp merging operation is within the realm of classical queueing theory.

Thus, the literature of queueing systems can be used to analyze the 'performance' of an entrance ramp. Unfortunately, the explicit solutions of queueing systems are known only for some forms of f_q and h_Q. When the explicit solutions are not known, simulation techniques are usually used to obtain approximate solutions.

The traffic engineering literature indicates that the time gaps of the traffic stream in the outside freeway lane can be characterized by the Erlang distribution. In this situation, a computer program that generates gamma variates is needed either to evaluate the performance of an entrance ramp or to properly design entrance ramps.

Example 6 - Roadside Vehicle Inspection Program

Consider the design of a roadside safety or emission level inspection program in accordance with the following two objectives: first, maximize the number of vehicles inspected in a fixed period for the funding available; second, minimize the delay time of each vehicle inspected.

In order to design a roadside inspection program that satisfies the above requirements, we must know the following:

a) inspection procedures
b) distribution of inspection times
c) sampling procedures (how vehicles are selected from the traffic stream for inspection)
d) estimated efficiency of each inspection station.

This information would be used to compute the following three probabilities:

P_1: For a roadway of traffic volume q and a given sampling (selecting) procedure, the probability that there are more than N vehicles entering the inspection station within a fixed time period.

P_2: Conditioned on more than N vehicles entering the inspection station within a fixed time period, the probability that at least N vehicles can be inspected with one inspection line, and with two or more parallel lines.

P_3: If more than N vehicles are inspected within a fixed period, the probability that the number of vehicles delayed more than W minutes is no more than L.

The product of the above three probabilities is the probability that an inspection station is able to inspect at least N vehicles within a fixed period and the number of vehicles delayed more than W minutes is no more than L. This information along with cost information can then be used, with the help of an optimization procedure, to design an 'optimal' roadside inspection program.

The problem at this point turns on the computation of the above three probabilities. Although this problem is within the realm of queueing theory, obtaining an explicit solution depends on the traffic pattern, sampling procedure, inspection discipline, and the distribution of inspection times. Most likely, this problem could only be solved by simulation. As noted before, many traffic variables are gamma distributed. Thus, a computer program that generates gamma variates will be an essential tool in designing this program.

We have used six practical examples in highway research and testing to demonstrate the usefulness of a computer program that generates normal and gamma variates. In the remaining sections we shall discuss the algorithm for writing such a computer program.

Gamma, Normal, and Chi-Square Distributions

The gamma distributions with parameters α and β is defined as

$$G(y;\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} \int_0^y x^{\alpha-1} e^{-x/\beta}\, dx \tag{15}$$

where $\alpha > 0$, $\beta > 0$ and $y \geq 0$ and where $\Gamma(\alpha)$ represent the well-known gamma function defined as

$$\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{-x}\, dx \tag{16}$$

When α is an integer, $G(y; \alpha, \beta)$ defined in Eq. (15) can be expressed as the sum of the finite series:

$$G(y, \alpha, \beta) = 1 - e^{-y/\beta} \sum_{n=0}^{\alpha-1} \frac{(y/\beta)^n}{n!} \tag{17}$$

When α is not an integer, we may use the Taylor expansion of the exponential function, and express $G(y; \alpha, \beta)$ as the sum of an infinite series:

$$G(y, \alpha, \beta) = \sum_{n=1}^{\infty} a_n \tag{18}$$

where

$$a_n = (-1)^{n+1} \frac{1}{\Gamma(\alpha)\,\Gamma(n)} \frac{(y/\beta)^{\alpha+n-1}}{\alpha+n-1}, \; n = 1, \ldots \tag{19}$$

Define M^* to be the least integer that is greater than $y/\beta + 1$. That is,

$$M^* = \mathrm{Min}\left\{n; n \geq y/\beta + 1\right\} \tag{20}$$

It can be shown that $\left\{a_n; n \geq M^*\right\}$ is an alternating sequence with the following properties:

$$|a_{n+1}| \leq |a_n| \text{ for every } n \geq M^* \tag{21}$$

and

$$a_n \to 0 \text{ as } n \to \infty \tag{22}$$

It is well-known that any alternating sequence with properties (21) and (22) implies the following useful inequality.

$$\left|\sum_{n=1}^{\infty} a_n - \sum_{n=1}^{N} a_n\right| \leq \left|a_{N+1}\right| \text{ for every } N \geq M^* \tag{23}$$

Inequality (23) states that $\hat{G}(y; \alpha, \beta)$ defined as the sum of the first N terms of the sequence $\left\{a_n\right\}$ is an approximation to $G(y; \alpha, \beta)$ with error no more than the absolute value of a_{N+1}. Thus, if ϵ is an acceptable approximation error and if N is chosen to be the larger integer of M^*, defined in Eq. (20) and k^*, defined as

$$K^* = \mathrm{Min}\left\{n; \frac{1}{\Gamma(\alpha)\,\Gamma(n)} \frac{(y/\beta)^{\alpha+n}}{\alpha+n} \leq \epsilon\right\} \tag{24}$$

then, $\hat{G}(y; \alpha, \beta) = \sum_{n=1}^{N} a_n$ is an approximation to $G(y; \alpha, \beta)$ with error no more than ϵ.

The chi-square distribution is a special case of the gamma distribution. More precisely, G(y; n/2, 2) is the chi-square distribution with n degrees of freedom.

It is well-known that, for any $y \geq 0$,

$$\int_0^y \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx = \tfrac{1}{2} G(y^2; 0.5, 2) \tag{25}$$

Thus, the distribution function N(y) defined as,

$$N(y) = \begin{cases} 0.5 - \tfrac{1}{2} G(y^2; 0.5, 2) & , y \leq 0 \\ 0.5 + \tfrac{1}{2} G(y^2; 0.5, 2) & , y > 0 \end{cases} \tag{26}$$

is the standard normal distribution.

This completes the computational formula for either computing or approximating the gamma, normal, and chi-square distributions. In the next two sections, we shall discuss the iterative procedures needed in solving the equation $P = G(y; \alpha, \beta)$ where P is a specified probability.

An Iterative Procedure for Finding a Gamma Variate

For any $0 \leq p < 1$, we are interested in solving the equation

$$P = G(y; \alpha, \beta) \tag{27}$$

One method of solving Eq. (27) is called the 'iterative procedure' which can be outlined in the following two steps: 1) obtain the initial approximation of the solution, and 2) repeatedly improve the approximations until the desired degree of accuracy is obtained. The details follow.

Step 1) Obtaining the Initial Approximation of the Solution

Denote μ_Y and σ_Y^2 to be the mean and variance of the non-negative random variable Y, respectively. The Cantelli Inequality states that for any $\lambda \geq 0$,

$$P_r(Y \leq \mu_Y - \lambda) \leq \frac{\sigma_Y^2}{\sigma_Y^2 + \lambda^2} \tag{28}$$

If Y has the distribution $G(y; \alpha, \beta)$, $\mu_Y = \alpha\beta$ and $\sigma_Y^2 = \alpha\beta^2$. By setting $P = \frac{\alpha\beta^2}{\alpha\beta^2 + \lambda^2}$, we obtain $\lambda = \sqrt{\alpha\beta^2(1-P)/P}$

Define

$$y_{11} = \text{Max}\left\{0, \alpha\beta - \sqrt{\alpha\beta^2(1-P)/P}\right\} \tag{29}$$

It is apparent that $y_{11} \leq y^*$; the true solution of Eq. (27).

By the simple transformation $W = y/\beta$, we find that $G(y;\alpha,\beta) = G(y/\beta;\alpha, 1)$. Thus, by using the decreasing property of the negative exponential function, we have the following inequality:

$$G(y;\alpha,\beta) = \frac{1}{\Gamma(\alpha)}\int_0^{y/\beta} x^{\alpha-1} e^{-x} dx \leq \frac{1}{\Gamma(\alpha)}\int_0^{y/\beta} x^{\alpha-1} dx = \frac{(y/\beta)^{\alpha}}{\Gamma(\alpha+1)} \qquad (30)$$

If we let y_{12} be the solution of $p = (y/\beta)^{\alpha}/\Gamma(\alpha+1)$, that is,

$$y_{12} = \beta\left[P\ \Gamma(\alpha+1)\right]^{1/\alpha} \qquad (31)$$

Then, again $y_{12} \leq y^*$. Now, define y_1 to be the maximum of y_{11} and y_{12} defined in Eq. (29) and Eq. (34), respectively. The y_1 is a underestimate of the true solution y^* and is to be used as the initial estimate of y^* in Step 2.

Step 2) To Improve the Initial Estimate

Let $P_1 = G(y_1;\ \alpha,\beta)$. Again, by using the decreasing property of the negative exponential function, the following inequality holds for any $y \geq y_1$.

$$G(y;\alpha,\beta) - P_1 = \frac{1}{\Gamma(\alpha)}\int_{(y_1/\beta)}^{y/\beta} x^{\alpha-1} e^{-x} dx < \frac{1}{\Gamma(\alpha+1)} e^{-y_1/\beta}\left[(y/\beta)^{\alpha} - (y_1/\beta)^{\alpha}\right] \qquad (32)$$

Set the right-hand side of Inequality (32) equal to $p - p_1$ and denote y_2 to be its solution, we have

$$y_2 = \beta\left[(P-P_1)\,\Gamma(\alpha+1)\,e^{y_1/\beta} + (y_1/\beta)^{\alpha}\right]^{1/\alpha} \qquad (33)$$

It is easy to check that $y_1 < y_2 \leq y^*$. Now, treating y_2 as the initial estimate of y^* and repeating the above procedures, we obtain

$$y_3 = \beta\left[(P-P_2)\,\Gamma(\alpha+1)\,e^{y_2/\beta} + (y_2/\beta)^{\alpha}\right]^{1/\alpha} \qquad (34)$$

By repeating the above iterative procedure we construct an increasing sequence $\{y_i\}$ with the following properties:

a) $y_i \leq y^*,\ i \geq 1$

b) $y_{i+1} = \beta\left[(P-P_i)\,\Gamma(\alpha+1)\,e^{y_i/\beta} + (y_i/\beta)^{\alpha}\right]^{1/\alpha}$, where $P_i = G(y_i;\alpha,\beta)$

c) $y_i \to y^*$ as $i \to \infty$

The number of iterations required to obtain y_N such that $y^* - y_N \leq \epsilon$ can be substantially reduced if the initial approximation is good. Obtaining a good initial approximation is the subject of the next section.

An Approximation Function of the Inverse of $G(y_i;\ \alpha, \beta)$

Let $p = G(y;\ \alpha, \beta)$ and $y = H(p;\ \alpha, \beta)$ for $0 \leq p < 1$. Since $G(y;\ \alpha, \beta) = G(y/\beta;\ \alpha,\ 1)$, we have the following relation:

$$H(P; \alpha, \beta) = \beta H(P; \alpha, 1) \tag{35}$$

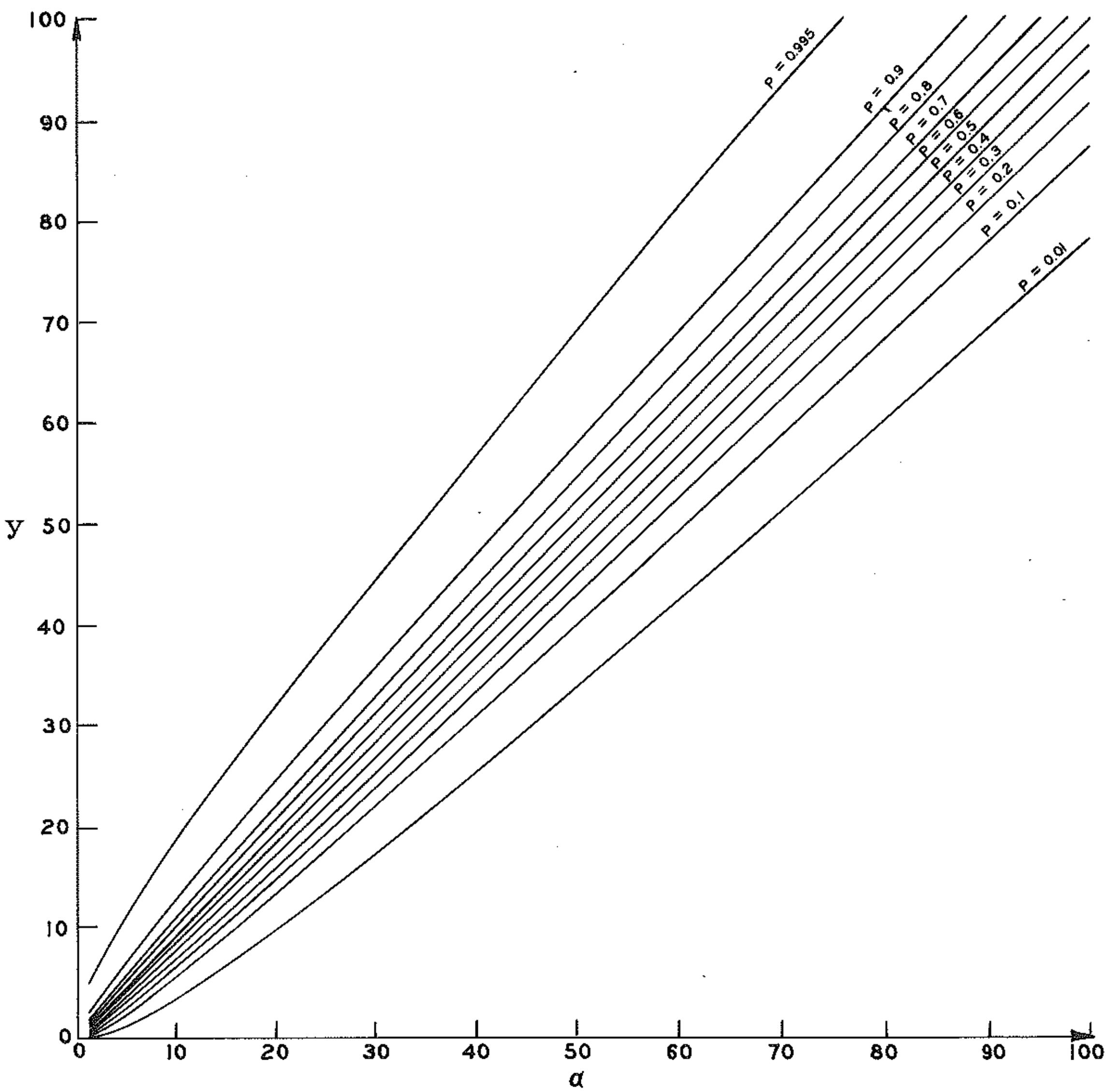

Figure 6. The relationship between the gamma variate and the parameter α when $\beta = 1$.

In order to find the approximation function of H(p; α, 1), a computer program was written based on techniques presented in the previous two sections to generate data consisting of various values of p, α and the corresponding y. A partial data set is presented in Figure 6. The functional relation between y and α is almost linear when P is close to 0.5 and approaches an s-curve when p is some distance from 0.5. Thus, for each fixed p, $y = a + b\alpha^c$ could well describe the relation between y and α. A non-linear curve-fitting computer program was then used to fit the generated data to determine the best coefficients for 17 selected p-values. The results are presented in Table 2 for $0 < \alpha \leqslant 20$ and Table 3 for $\alpha > 20$.

TABLE 2

THE FITTED COEFFICIENTS a, b, AND c OF THE EQUATION $y = a + b\alpha^c$ FOR $0 < \alpha \leqslant 20$

P	a	b	c	Standard Error
0.010	-0.368840	0.193453	1.364790	0.0728
0.020	-0.401263	0.244240	1.311010	0.0679
0.030	-0.419850	0.281988	1.277970	0.0644
0.040	-0.434117	0.313220	1.253970	0.0614
0.050	-0.442727	0.340560	1.234840	0.0590
0.100	-0.467346	0.446127	1.174570	0.0498
0.200	-0.470211	0.605737	1.106750	0.0320
0.300	-0.433872	0.736681	1.064950	0.0219
0.400	-0.379534	0.863192	1.031340	0.0105
0.500	-0.310705	0.993767	1.001810	0.0029
0.600	-0.195109	1.127820	0.976080	0.0066
0.700	-0.046760	1.283350	0.950197	0.0146
0.800	0.194695	1.469830	0.923979	0.0231
0.900	0.642609	1.738140	0.892864	0.0327
0.950	1.128570	1.964630	0.871034	0.0388
0.975	1.632310	2.164600	0.854310	0.0451
0.995	2.863550	2.556870	0.826947	0.0509

TABLE 3
THE FITTED COEFFICIENTS a, b, AND c FOR THE EQUATION
$y = a + b\alpha^c$ FOR $\alpha > 20$

P	a	b	c	Standard Error
0.010	-2.945360	0.521921	1.096020	0.0277
0.020	-2.762260	0.568521	1.083100	0.0243
0.030	-2.625400	0.599105	1.075250	0.0219
0.040	-2.523630	0.623322	1.069290	0.0212
0.050	-2.435650	0.643476	1.064520	0.0191
0.100	-2.082150	0.714516	1.048970	0.0144
0.200	-1.564780	0.805747	1.031340	0.0098
0.300	-1.150270	0.876921	1.018940	0.0063
0.400	-0.743587	0.939213	1.009030	0.0042
0.500	-0.329996	1.000000	1.000000	0.0000
0.600	0.118335	1.062230	0.991392	0.0036
0.700	0.626079	1.132350	0.982270	0.0057
0.800	1.293260	1.215700	0.972270	0.0084
0.900	2.296960	1.337770	0.958865	0.0126
0.950	3.221920	1.442310	0.948437	0.0157
0.975	4.088400	1.536170	0.939768	0.0174
0.995	5.988710	1.725150	0.924052	0.0222

By plotting a, b, and c versus p in Figure 7, we see that it is feasible to express coefficients as some functions of p. Denote a_p, b_p, and c_p as coefficients of the equation corresponding to p. Thus, we have

$$y = \beta(a_p + b_p \alpha^{c_p}), \alpha > 0, \beta > 0 \text{ and } 0 \leq p < 1 \qquad (36)$$

Since our goal is to obtain a good initial approximation of the solution and it is also practical to build Tables 2 and 3 into a computer program,

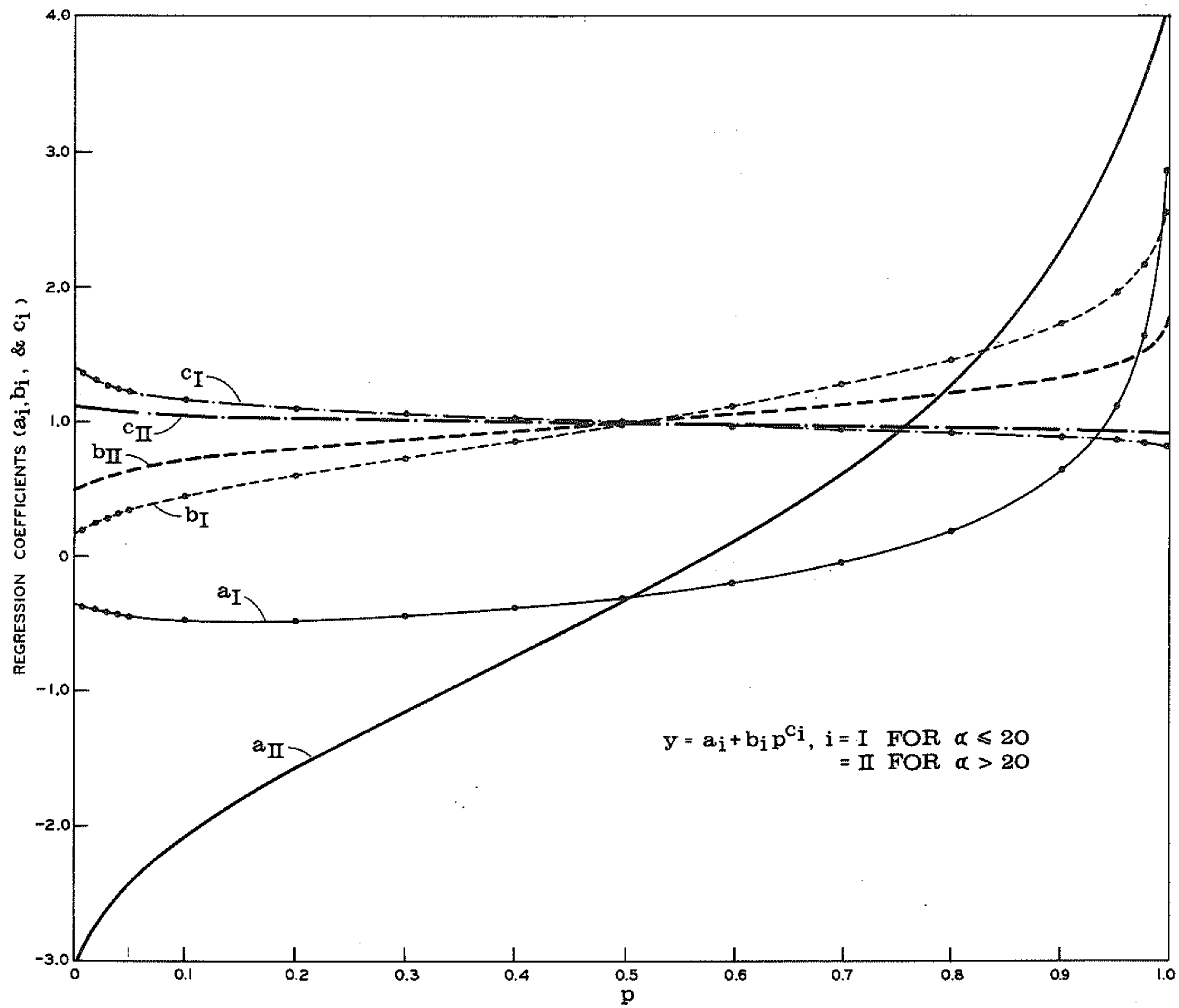

Figure 7. The relationships between each regression coefficient and the gamma probability.

we present the following method for determining the coefficients of Eq. (36).

Observe that any segment of curves in Figure 7 can be well fitted by a polynomial function of second order. That is, the following simultaneous equations are locally valid:

$$a_p = t_1 + t_2 p + t_3 p^2 \tag{37}$$

$$b_p = w_1 + w_2 p + w_3 p^2 \tag{38}$$

and

$$c_p = v_1 + v_2 p + v_3 p^2 \tag{39}$$

The procedures for determining t_i, W_i and V_i are as follows:

a) Find three consecutive p's either from Table 2 or Table 3, say p_1, p_2 and p_3, covering p. That is, $p_1 \leqslant p \leqslant p_3$ and $p_1 \leqslant p_2 \leqslant p_3$. In the case that $p < 0.01$ or $p > 0.995$, we choose the first or last three p's, respectively.

b) Use Table 1 or Table 2, depending on the value of α, to obtain (p_1, a_{p_1}), (p_2, a_{p_2}) and (p_3, a_{p_3}). These three points are then used to determine a polynomial equation of second order. That is, $t_i = \det(A_i)$, $i = 1$, 2, and 3, where A is the matrix defined in Eq. (40), A_i is the matrix obtained from A by replacing the i^{th} column with $(a_{p_1}, a_{p_2}, a_{p_3})^t$, the superscript t denotes the transpose of the vector or matrix, det denotes the determinant of a square matrix. W_i and V_i are obtained similarly.

$$A = \begin{bmatrix} 1 & p_1 & p_1^2 \\ 1 & p_2 & p_2^2 \\ 1 & p_3 & p_3^2 \end{bmatrix} \tag{40}$$

c) Once t_i, W_i and V_i are determined, a_p, b_p, and c_p are then determined from Eqs. (37) through (30). Consequently, y is then determined from Eq. (36).

The y obtained by the above procedure is very close to the solution of the equation $p = G(y; \alpha, \beta)$. Denote y_1 to be the y-value obtained from these procedures and $p_1 = G(y_1; \alpha, \beta)$. If $p_1 < P$, we use y_1 as the initial approximation to find the solution by the iterative method described in the preceeding section. If $P_1 > P$, then

$$P_1 - P = \frac{1}{\Gamma(\alpha)} \int_{y/\beta}^{y_1/\beta} x^{\alpha-1} e^{-x}\, dx \geqslant \frac{1}{\Gamma(\alpha+1)} e^{-y_1/\beta} \left[(y_1/\beta)^{\alpha} - (y/\beta)^{\alpha} \right] \qquad (41)$$

Set the right-hand side of Eq. (41) equal to $P_1 - P$ and denote y_2 to be its solution, that is,

$$y_2 = \beta \left[(y_1/\beta)^{\alpha} - (P_1 - P)\, \Gamma(\alpha+1)\, e^{y_1/\beta} \right]^{1/\alpha} \qquad (42)$$

then, it can be shown that $y_2 \leqslant y \leqslant y_1$. Moreover, y_2 will also be very close to the solution. In this situation, the curve between y_2 and y_1 is almost linear. Define

$$y_3 = (y_1 - y_2) \frac{P - G(y_2; \alpha, \beta)}{G(y_1; \alpha, \beta) - G(y_2; \alpha, \beta)} \qquad (43)$$

Since $G(y; \alpha, \beta)$ is an increasing function of y, $y_2 \leqslant y_3 \leqslant y_1$. Repeating the above linear interpolation by using either (y_2, y_3) or (y_3, y_1) depending on the location of y_3, we would be able to obtain the approximation to the solutions within the desired accuracy in very few interations.

APPENDIX

A FORTRAN COMPUTER PROGRAM FOR COMPUTING PROBABILITY AND GENERATING RANDOM VARIATES FOR THE GAMMA, NORMAL AND CHI-SQUARE DISTRIBUTIONS

Based on the techniques presented earlier in this report, we present a FORTRAN Computer Program which is designed to perform the following tasks:

a) To compute the probability that a gamma, normal, or chi-square random variable x lies between θ_1 and θ_2. That is, to compute,

$$P = P_r(\theta_1 \leq X \leq \theta_2) = \int_{\theta_1}^{\theta_2} f(x)dx \qquad \text{(A-1)}$$

where f is the gamma density function with parameters α and β, the normal density function with parameters μ and σ^2, or the chi-square density function with N degrees of freedom.

b) To find the solution of the equation,

$$P = P_r(x \leq \theta) \qquad \text{(A-2)}$$

for a given probability P, where x is a gamma, normal, or chi-square random variable.

This program has three subroutines: GAMM, BEGIN, and SOLVE. The subroutine GAMM will compute the probability that a gamma random variable is less than or equal to a given non-negative quantity θ. The approximation error is no greater than 0.0000000000001. The subroutine BEGIN will obtain the initial approximation of Eq. (A-2) based on techniques presented in the last section of the report. The subroutine SOLVE will improve the approximations until the desired accuracy is achieved. The desired accuracy is set at 0.0000001. That is, if $|\hat{P} - P| \leq 0.0000001$, where $\hat{P} = Pr(x \leq \hat{y})$, $\hat{y}$ is taken as the solution of Eq. (A-2). The convergence criterion can be easily changed in this subroutine. Thus, users are encouraged to use different convergence criteria to fit their needs.

This program can handle many sets of computations defined in a) and b) in one run. Each set is specified by two data cards or records. The input is diagrammed in Figure A-1 to show the format and logic used. Definitions are presented in Table A-1. The structure of the read statement and the format used are given so that users can construct a data file to fit their problem.

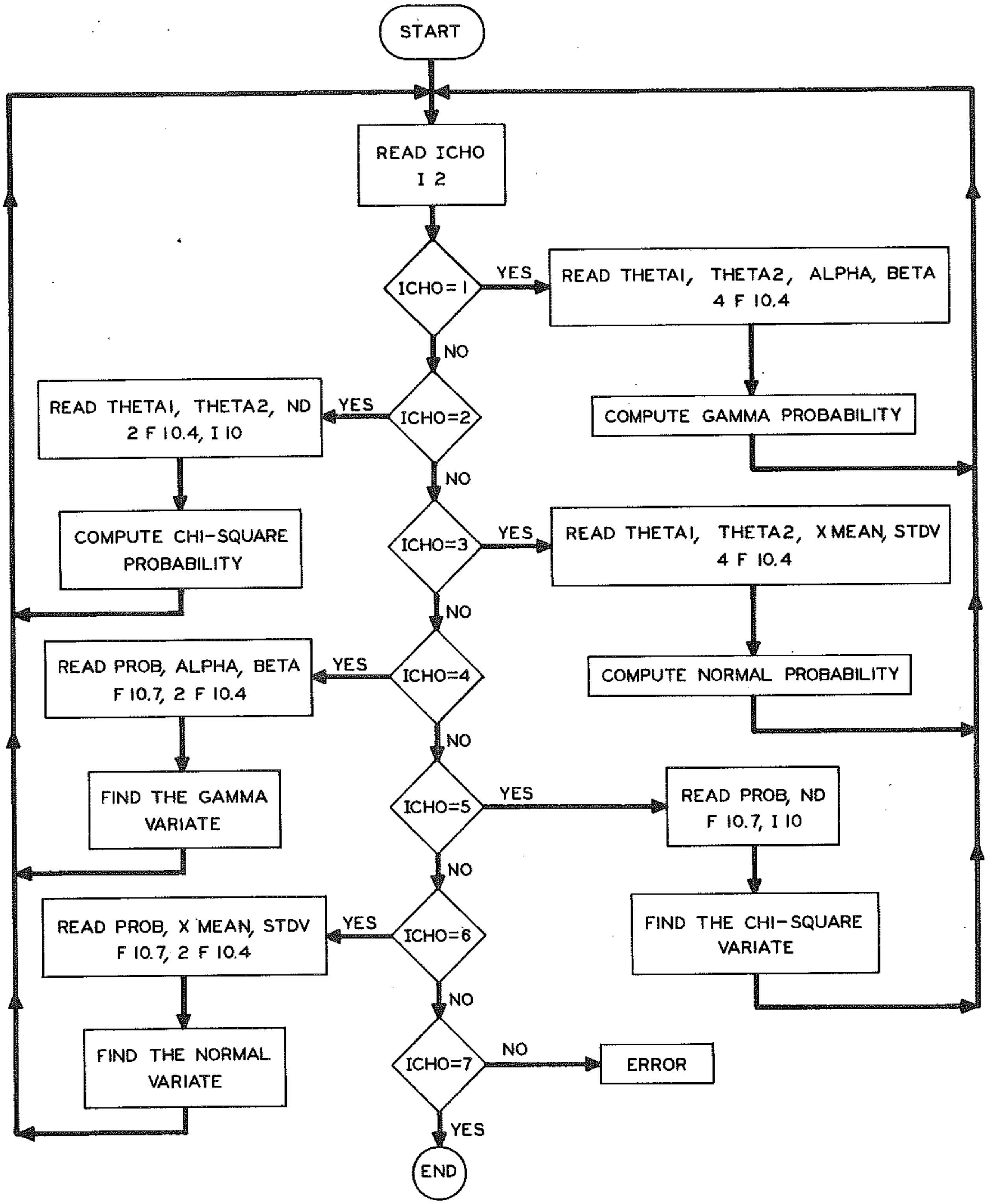

Figure A-1. Flow chart of the input systems.

TABLE A-1
DEFINITIONS OF VARIABLES

Name	Definition
ICHO	Types of Computation 1 - for computing the gamma probability 2 - for computing the chi-square probability 3 - for computing the normal probability 4 - for finding the gamma variate 5 - for finding the chi-square variate 6 - for finding the normal variate 7 - to stop the computer run
ALPHA BETA	The parameters of the gamma distribution
ND	The parameter (degrees of freedom) of the chi-square distribution
XMEAN	The mean of the normal distribution
THETA 1	The lower limit of integration
THETA 2	The upper limit of integration
PROB	The given probability
STDV	The standard deviation of the normal distribution

```
      FILE  1=WHK/GCN/DATA,UNIT=DISK,BLOCKING=15,RECORD=14
      FILE  2=OUT,UNIT=PRINTER
            DIMENSION PP(18),COEF(2,17,3)
            COMMON PP,COEF
            DATA (PP(I),I=1,18)/0.01,0.02,0.03,0.04,0.05,
     -      0.1,0.2,0.3,0.4,0.5,0.6,0.7,0.8,0.9,
     -      0.95,0.975,0.995,1.0/
            DATA ((COEF(1,I,J),J=1,3),I=1,17)/
     -      -0.368840, 0.193453, 1.364790,-0.401263, 0.244240, 1.311010,
     -      -0.419850, 0.281988, 1.277970,-0.434117, 0.313220, 1.253970,
     -      -0.442727, 0.340560, 1.234840,-0.467346, 0.446127, 1.174570,
     -      -0.470211, 0.605737, 1.106750,-0.433872, 0.736681, 1.064950,
     -      -0.379534, 0.863192, 1.031340,-0.310705, 0.993767, 1.001810,
     -      -0.195109, 1.127820, 0.976080,-0.046760, 1.283350, 0.950197,
     -      0.194695, 1.469830, 0.923979, 0.642609, 1.738140, 0.892864,
     -      1.128570, 1.964630, 0.871034, 1.632310, 2.164600, 0.854310,
     -      2.863550, 2.556870, 0.826947/
            DATA ((COEF(2,I,J),J=1,3),I=1,17)/
     -      -2.945360, 0.521921, 1.096020,-2.762260, 0.568521, 1.083100,
     -      -2.625400, 0.599105, 1.075250,-2.523630, 0.623322, 1.069290,
     -      -2.435650, 0.643476, 1.064520,-2.082150, 0.714516, 1.048970,
     -      -1.564780, 0.805747, 1.031340,-1.150270, 0.876921, 1.018940,
     -      -0.743587, 0.939213, 1.009030,-0.329996, 1.000000, 1.000000,
     -      0.118335, 1.062230, 0.991392, 0.626079, 1.132350, 0.982270,
     -      1.293260, 1.215700, 0.972270, 2.296960, 1.337770, 0.958865,
     -      3.221920, 1.442310, 0.948437, 4.088400, 1.536170, 0.939768,
     -      5.988710, 1.725150, 0.924052/
            NSET=0
        1   CONTINUE
            NSET=NSET+1
            WRITE(2,7) NSET
            READ(1,9) ICHO
            GO TO (100,200,300,400,500,600,9999),ICHO
      100   READ(1,105)THETA1,THETA2,ALPHA,BETA
            WRITE(2,120)ALPHA,BETA
            GO TO 204
      200   READ(1,205) THETA1,THETA2,ND
            WRITE(2,202)ND
            ALPHA=ND/2.
            BETA=2.
      204   CONTINUE
            IF(THETA1 .GE. 0.) GO TO 220
            WRITE(2,210)
            GO TO 9999
      220   CONTINUE
            IF(THETA1 .LE. THETA2) GO TO 240
            WRITE(2,239)
            GO TO 9999
      240   CONTINUE
            CALL GAMM(THETA1,ALPHA,BETA,PROB1)
            CALL GAMM(THETA2,ALPHA,BETA,PROB2)
            PROB=PROB2-PROB1
            WRITE(2,299) THETA1,THETA2,PROB
            GO TO 1
      300   READ(1,105) THETA1,THETA2,XMEAN,STDV

            WRITE(2,302) XMEAN,STDV
```

```
      IF(THETA1 .LE. THETA2) GO TO 304
      WRITE(2,239)
      GO TO 9999
  304 CONTINUE

      Z1=(THETA1-XMEAN)/STDV
      Z2=(THETA2-XMEAN)/STDV
      ZZ1=Z1**2
      ZZ2=Z2**2
      ALPHA=0.5
      BETA=2.
      CALL GAMM(ZZ1,ALPHA,BETA,PROB1)
      CALL GAMM(ZZ2,ALPHA,BETA,PROB2)
      PROB=PROB1+PROB2
      IF(Z2 .LE. 0.) PROB=PROB1-PROB2
      IF(Z1 .GE. 0.) PROB=PROB2-PROB1
      PROB=PROB/2.
      WRITE(2,299) THETA1,THETA2,PROB
      GO TO 1
  400 READ(1,405) PROB,ALPHA,BETA
      WRITE(2,120) ALPHA,BETA
      GO TO 518
  500 READ(1,505) PROB,ND
      WRITE(2,202) ND
      ALPHA=ND/2.
      BETA=2.
  518 CONTINUE
      CALL BEGIN(PROB,ALPHA,BETA,T1,T2,P1,P2)
      MTH=1
      IF(T2 .GE. 0.) MTH=2
      CALL SOLVE(MTH,T1,T2,P1,P2,PROB,ALPHA,BETA,THETA)
      WRITE(2,520) PROB,THETA
      GO TO 1
  600 READ(1,405) PROB,XMEAN,STDV
      WRITE(2,302) XMEAN,STDV
      ALPHA=0.5
      BETA=2.0
      IF(PROB .LT. 0.5) PRB=2.*(0.5-PROB)
      IF(PROB .GE. 0.5) PRB=2.*(PROB-0.5)
      CALL BEGIN(PRB,ALPHA,BETA,T1,T2,P1,P2)
      MTH=1
      IF(T2 .GE. 0.) MTH=2
      CALL SOLVE(MTH,T1,T2,P1,P2,PRB,ALPHA,BETA,THETA)
      TTH=THETA**0.5
      IF(PROB .LT. 0.5) THETA=XMEAN-STDV*TTH
      IF(PROB .GE. 0.5) THETA=XMEAN+STDV*TTH
      WRITE(2,520) PROB,THETA
      GO TO 1
 9999 CONTINUE
      STOP
C-########## FORMATS ##########
    7 FORMAT (/,2X,9H##### SET,I3,1X,5H#####,/)

    9 FORMAT (I2)

  105 FORMAT (4F10.4)
  120 FORMAT (2X,22HGAMMA DIST WITH ALPHA=,E15.6,/,19X,
```

```
     -      5HBETA=,E15.6)
      202   FORMAT (2X,26HCHI-SQUARE DIST WITH D.F. ,I6)
      205   FORMAT (2F10.4,I10)
      210   FORMAT (/,2X,29HTHETA1 SHOULD BE NON-NEGATIVE,/)
      239   FORMAT (/,2X,36HTHETA2 SHOULD BE GREATER THAN THETA1,/)
      299   FORMAT (/,2X,5HPROB(,E10.4,3H , ,E10.4,3H )=,F30.25,/)
      302   FORMAT (2X,22HNORMAL DIST WITH MEAN=,E15.6,/,19X,
     -      5HSTDV=,E15.6)
      405   FORMAT (F10.7,2F10.4)
      505   FORMAT (F10.7,I10)
      520   FORMAT (/,2X,5HPROB=,E15.6,/,2X,
     -      19HTHE RANDOM VARIATE=E15.6)
     C- ##############################
     C-
            END
     C-
     C- ##### SUBROUTINE BEGIN #####
     C-
            SUBROUTINE BEGIN(PROB,ALPHA,BETA,THETA1,THETA2,PROB1,PROB2)
            DIMENSION PP(18),COEF(2,17,3),CONST(3),AA(3,3),BB(3,3),U(4),
     -      EU(3)
            COMMON PP,COEF
     C-
     C-##########
     C-   SUBROUTINE FOR COMPUTING THE INITIAL THETA OR THETA INTERVAL
     C-   FOR THE ITERATIVE PROCEDURES OF FINDING THE THETA SUCH THAT
     C-   THE PROBABILITY OF A GAMMA R.V. (WITH PARAMETERS ALPHA &
     C-   BETA) LESS THAN OR EQUAL TO THETA IS PROB.
     C-
     C-   THE FOLLOWING FIGURES ARE COEFFICIENTS A,B & C OF THE EQUATION
     C-           THETA=A+B*ALPHA**C
     C-   FOR EACH P AND TWO SETS OF ALPHA RANGES. THESE FIGURES SHOULD
     C-   BE DEFINED IN THE MAIN PROGRAM OR THE CALLING SUBROUTINE. THE
     C-   FIRST FIGURE IS P-VALUE. THE NEXT THREE FIGURES ARE A,B,C FOR
     C-   THE CORRESPONDING P AND ALPHA INBETWEEN 1 & 20. THE LAST THREE
     C-   ARE FOR THE ALPHA IN THE RANGE OF 20 TO 100.
     C-
     C- 0.010,-0.368840, 0.193453, 1.364790,-2.945360, 0.521921, 1.096020,
     C- 0.020,-0.401263, 0.244240, 1.311010,-2.762260, 0.568521, 1.083100,
     C- 0.030,-0.419850, 0.281988, 1.277970,-2.625400, 0.599105, 1.075250,
     C- 0.040,-0.434117, 0.313220, 1.253970,-2.523630, 0.623322, 1.069290,
     C- 0.050,-0.442727, 0.340560, 1.234840,-2.435650, 0.643476, 1.064520,
     C- 0.100,-0.467346, 0.446127, 1.174570,-2.082150, 0.714516, 1.048970,
     C- 0.200,-0.470211, 0.605737, 1.106750,-1.564780, 0.805747, 1.031340,
     C- 0.300,-0.433872, 0.736681, 1.064950,-1.150270, 0.876921, 1.018940,
     C- 0.400,-0.379534, 0.863192, 1.031340,-0.743587, 0.939213, 1.009030,
     C- 0.500,-0.310705, 0.993767, 1.001810,-0.329996, 1.000000, 1.000000,
     C- 0.600,-0.195109, 1.127820, 0.976080, 0.118335, 1.062230, 0.991392,
     C- 0.700,-0.046760, 1.283350, 0.950197, 0.626079, 1.132350, 0.982270,
     C- 0.800, 0.194695, 1.469830, 0.923979, 1.293260, 1.215700, 0.972270,
     C- 0.900, 0.642609, 1.738140, 0.892864, 2.296960, 1.337770, 0.958865,
     C- 0.950, 1.128570, 1.964630, 0.871034, 3.221920, 1.442310, 0.948437,
     C- 0.975, 1.632310, 2.164600, 0.854310, 4.088400, 1.536170, 0.939768,
     C- 0.995, 2.863550, 2.556870, 0.826947, 5.988710, 1.725150, 0.924052,
     C-###########
     C-
            THETA2=-99.
```

```
      GMEAN=ALPHA*BETA
      GVAR=GMEAN*BETA
      ALPHA1=ALPHA+1.0
      GALPHA=ALGAMA(ALPHA1)
C-##### VARIOUS METHODS ARE USED TO OBTAIN THE BEST INITIAL
C-       ESTIMATE OF THE THETA FOR THE ITERATIVE PROCEDURE
C-##### METHOD(1)---CANTELLI INEQUALITY
      T0=0.
      W1=GVAR*(1.-PROB)/PROB
      T1=GMEAN-SQRT(W1)
C-##### METHOD(2)---MONOTONIC PROPERTY OF THE EXPONENTIAL FUNCTION
      W1=(ALOG(PROB)+GALPHA)/ALPHA
      T2=BETA*EXP(W1)
      THETA1=AMAX1(T0,T1,T2)
C-##### METHOD(3)---REGRESSION EQUATION
      ISET=1
      IF (ALPHA .GT. 20.0) ISET=2
      DO 20 J=1,18
      IF (PROB .LE. PP(J)) GO TO 25
  20  CONTINUE
  25  JJ=J-1
      IF (JJ .EQ. 0)JJ=JJ+1
      IF (JJ .EQ. 16)JJ=JJ-1
      IF (JJ .EQ. 17)JJ=JJ-2
C-##### JJ IS THE BEGINNING INDEX OF THREE POINTS USED TO DETERMINE
C-        A POLYNOMIAL EQUATION
      DO 30 I=1,3
      AA(I,1)=1.0
      DO 35 J=2,3
      AA(I,J)=PP(JJ+I-1)**(J-1)
  35  CONTINUE
  30  CONTINUE
      DO 37 IA=1,3
      DO 38   I=1,3
      JI=JJ+I-1
      CONST(I)=COEF(ISET,JI,IA)
      DO 39 J=1,3
      BB(I,J)=AA(I,J)
  39  CONTINUE
  38  CONTINUE
      JP=4
  40  CONTINUE
      IF (JP .EQ. 4) GO TO 100
      DO 50 I=1,3
      BB(I,JP)=CONST(I)
  50  CONTINUE
      J=JP+1
      IF (J .GE. 4) GO TO 100
      DO 60 I=1,3
      BB(I,J)=AA(I,J)
  60  CONTINUE
C-##### COMPUTE DETERMINANT OF A 3 X 3 MATRIX
 100  CONTINUE
      U(JP)=0.
      DO 110 I=1,3
      AU=1.0
      DU=1.0
```

```
      DO 120 J=1,3
      J1=4-J
      K=I+J-1
      IF (K .GT. 3) K=K-3
      AU=AU*BB(K,J)
      DU=DU*BB(K,J1)
  120 CONTINUE
      U(JP)=U(JP)+AU-DU
  110 CONTINUE
      JP=JP-1
      IF (JP .GE. 1) GO TO 40
      DO 130 I=1,3
      U(I)=U(I)/U(4)
  130 CONTINUE
      EU(IA)=U(1)+U(2)*PROB+U(3)*PROB**2
   37 CONTINUE
      T3=(EU(1)+EU(2)*ALPHA**EU(3))*BETA
      IF(T3 .GT. THETA1)GO TO 135
      CALL GAMM(THETA1,ALPHA,BETA,PROB1)
      RETURN
  135 CONTINUE
C-##### CHECK TO SEE T3 IS AN OVERESTIMATE
      CALL GAMM(T3,ALPHA,BETA,PROB1)
      IF(PROB1 .GT. PROB) GO TO 142
      THETA1=T3
      RETURN
  142 CONTINUE
      THETA2=T3
      PROB2=PROB1
      AT=ALOG(PROB2-PROB)+GALPHA+T3/BETA-ALPHA*ALOG(T3/BETA)
      ATT=EXP(AT)
      IF(ATT .GT. 1.) ATT=1.
      T4=(1.-ATT)**(1./ALPHA)*T3
      IF(T4 .GT. THETA1) THETA1=T4
      CALL GAMM(THETA1,ALPHA,BETA,PROB1)
      RETURN
      END
C-
C- ##### SUBROUTINE SOLVE #####
C-
      SUBROUTINE SOLVE(MTH,T1,T2,P1,P2,PB,ALPHA,BETA,THETA)
      THETA=0.
      IF(PB .LE. 0.) GO TO 999
      ERROR=0.0000001
      IF(MTH .EQ. 2) GO TO 2000
      AL1=ALPHA+1.0
      G1=ALGAMA(AL1)
      THETA=T1
 1000 CONTINUE
      CHECK=ABS(PB-P1)
      IF(CHECK .LE. ERROR) GO TO 999
      AT=ALOG(PB-P1)+G1+THETA/BETA-ALPHA*ALOG(THETA/BETA)
      THETA=THETA*(EXP(AT)+1.0)**(1./ALPHA)
      CALL GAMM(THETA,ALPHA,BETA,P1)
      GO TO 1000
 2000 CONTINUE
      THETA=T1+(PB-P1)/(P2-P1)*(T2-T1)
```

```
      CALL GAMM(THETA,ALPHA,BETA,PP)
      CHECK=ABS(PB-PP)
      IF(CHECK .LE. ERROR) GO TO 999
      IF(PP .GE. PB) GO TO 2050
      T1=THETA
      P1=PP
      GO TO 2000
 2050 T2=THETA
      P2=PP
      GO TO 2000
  999 CONTINUE
      RETURN
      END
C-
C- ##### SUBROUTINE GAMM #####
C-
      SUBROUTINE GAMM(UL,ALPHA,BETA1,PROB)
      DOUBLE PRECISION AA,BB,CC,WORK,ABC,SUM,TERM,CHECK1,CHECK2
     ,DD,GG,HH
      IF(UL .GT. 0.) GO TO 2
      PROB=0.
      RETURN
    2 CONTINUE
      BETA=2.0
      THETA=UL*BETA/BETA1
      NL=ALPHA
      BL=ALPHA-NL
      AA=BL+1.
      BB=BETA
      CHECK1=0.000000000001
      ABC=THETA/BETA
      GG=DEXP(-ABC)
      IF(AA .GT. 1.) GO TO 4
      HH=ABC
      DD=1.
      WORK=1.-GG
      IF(NL .GT. 1) GO TO 100
      PROB=WORK
      RETURN
    4 CONTINUE
      DD=DGAMMA(AA)
      HH=ABC**AA
      WORK=1.0
      IF(THETA .LT. 52.0) GO TO 1055
      IF(THETA .GE. 58.0) GO TO 105
      IF(AA .LE. 1.2) GO TO 105
      IF(AA .LE. 1.3 .AND. THETA .GE. 53.0) GO TO 105
      IF(AA .LE. 1.6 .AND. THETA .GE. 54.0) GO TO 105
      IF(AA .LE. 1.7 .AND. THETA .GE. 55.0) GO TO 105
      IF(AA .LE. 1.9 .AND. THETA .GE. 56.0) GO TO 105
      IF(AA .LE. 2.0 .AND. THETA .GE. 58.0) GO TO 105
C-
C- ################################
C-    NMIN IS THE MINIMAL NUMBER OF TERMS IN TAYLOR SERIES NEEDED
C-       TO COMPUTE THE GAMMA INTEGRATION.
C- ################################
C-
```

```
 1055 CONTINUE
      NMIN=ABC+1.
      I=1
      WORK=HH/(DD*AA)
      SUM=WORK
 10   I=I+1
      CHECK2=I-1.
      CC=AA+I-1.
      TERM=(CC-1.)/(CC*CHECK2)*ABC*(-1.)
      WORK=WORK*TERM
      CHECK2=DABS(WORK)
      SUM=SUM+WORK
      IF(I .LT. NMIN .OR. CHECK2 .GE. CHECK1) GO TO 10
      WORK=SUM
C-
C- ##############################
C-   THE I-VALUE AT THIS STEP IS THE NUMBER OF TERMS IN TAYLOR
C-      SERIES USED TO APPROXIMATE THE GAMMA INTEGAL SO THAT THE
C-      MAXIMAL ERROR IS NO MORE THAN SPECIFIED QUANTITY-ERROR.
C- ##############################
C-
 105  CONTINUE
      IF(NL .EQ. 1) GO TO 200
      IF(NL .GT. 1) GO TO 100
      CC=ABC**(AA-1.)*GG/DD
      WORK=WORK+CC
      IF(WORK .GT. 1.) WORK=1.
      GO TO 200
 100  NL=NL-1
      DO 101 J=1,NL
      DD=DD*AA
      AA=AA+1.
      CC=HH*GG/DD
      WORK=WORK-CC
      HH=HH*ABC
 101  CONTINUE
 200  PROB=WORK
      RETURN
      END
```

Input Data

1009	1			
1019	0.0000	1.7855	5.0000	1.0000
1029	4			
1039	0.0350000	5.0000	1.0000	
1049	2			
1059	0.0000	1.1500	5	
1069	5			
1079	0.9000000	5		
1089	3			
1099	-15.0000	-1.3000	0.0000	1.0000
1109	3			
1119	1.1000	2.2000	0.0000	1.0000
1129	6			
1139	0.9678000	0.0000	1.0000	
1149	7			

Output

```
##### SET  1 #####

GAMMA DIST WITH ALPHA=    .500000E+01
                 BETA=    .100000E+01

PROB(0.          ,  .1785E+01 )=   0.0353675659294000000000000


##### SET  2 #####

GAMMA DIST WITH ALPHA=    .500000E+01
                 BETA=    .100000E+01

PROB=    .350000E-01
THE RANDOM VARIATE=    .178031E+01

##### SET  3 #####

CHI-SQUARE DIST WITH D.F.      5

PROB(0.          ,  .1150E+01 )=   0.0504167019808000000000000


##### SET  4 #####

CHI-SQUARE DIST WITH D.F.      5

PROB=    .900000E+00
THE RANDOM VARIATE=    .923636E+01

##### SET  5 #####

NORMAL DIST WITH MEAN=   0.
                 STDV=    .100000E+01

PROB(-.1500E+02 , -.1300E+01 )=   0.0968004845856000000000000


##### SET  6 #####

NORMAL DIST WITH MEAN=   0.
                 STDV=    .100000E+01

PROB( .1100E+01 ,  .2200E+01 )=   0.1217626134310000000000000


##### SET  7 #####

NORMAL DIST WITH MEAN=   0.
                 STDV=    .100000E+01

PROB=    .967800E+00
THE RANDOM VARIATE=    .184940E+01

##### SET  8 #####
```