THEORETICAL ANALYSIS OF A FLEXIBLE PAVEMENT MODEL

MICHIGAN HIGHWAY RESEARCH LABORATORY SECTION

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THEORETICAL ANALYSIS
OF A
FLEXIBLE PAVEMENT MODEL

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ABSTRACT

A method of analysis is presented for the investigation of developed stresses, pore pressures, and displacements in a pavement structure consisting of a flexible pavement supported on a water-saturated soil foundation.

A mathematical model of the pavement structure is constructed using ideal materials. The flexible pavement surface is replaced by an impermeable, linear viscoelastic, thin plate of infinite extent in the model. A porous elastic solid saturated with an incompressible, viscous fluid is used to simulate the saturated soil foundation. A uniform circular load is placed at the plate surface to approximate a vehicle wheel load. The mathematical model is analyzed to obtain an approximation of the deformation that would occur if the real pavement structure were subjected to a uniform circular load.

Analysis of the model is reduced to the solution of a linear initial-boundary value problem. The initial-boundary value problem is solved using iterated Laplace-Hankel transformations. The transformed solution images are inverted to obtain physically meaningful solutions to the problem using numerical methods. The Hankel transform inversion is performed approximately by numerical integration yielding values of the Laplace transform of the solution at discrete points. The Laplace transform data are then used to construct a generalized Fourier series approximation of the time-dependent solution.

A computer program was developed that utilized the inversion algorithm to invert the transformed solution functions. Stresses and displacement at any geometric point in the flexible pavement structure may be obtained as functions of time using the developed computer program.

Measured material properties were used to construct a numerical solution for the response to load of a hypothetical pavement structure. The results of the analysis are presented.

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LIST OF SYMBOLS

Roman Characters

A – Total surface area.

A_p – Pore area at surface.

a, a_1, a_2, a_3 – Physical constants of poro-elasticity.

b – Load radius.

B_1, B_2, B_3 – Functions of Laplace and Hankel variables.

c – Diffusion constant.

$C_1, C_2, C_3, C_4, C_5, C_6, C_7$ – Functions of Laplace and Hankel variables.

C_{mn} – Matrix of coefficients used in constructing orthonormal functions.

d_{ij} – Rate of deformation tensor for solid in mixture.

$D(t)$ – Plate stiffness function.

D – Plate geometric stiffness constant.

$d_1, d_1', d_2, d_2', d_3, d_3'$ – Functions of transform variables.

e_{ij} – Small strain tensor of solid.

$\hat{e}_r, \hat{e}_\theta, \hat{e}_z$ – Unit vectors.

E – Displacement generating function.

f – Porosity.

f_i – Acceleration vector of solid particles.

f_{ij} – Rate of deformation tensor of fluid.

F_i – Body force per unit mass on solid.

g_i – Acceleration vector of fluid particle.

G_i – Body force per unit mass on fluid particle.

h – Plate thickness.

$H(t)$ — Heaviside unit step function.

J_ν — Bessel function of first kind and ν order.

k — Function of Hankel variable, η.

K — Function of Hankel variable, η.

$\{\ell_m\}$ — Sequence of real numbers.

${}^{(1)}m, {}^{(2)}m$ — Intermediate variables.

M_x, M_y, M_{xy} — Moment resultants per unit length along edge of plate element.

n_k — Unit vector normal to arbitrary surface at a point.

$\bar{p}$ — An arbitrary scaler function.

p — The partial fluid pressure in the interacting mixture.

$p_0(t)$ — Scaler time dependent function.

q — Load pressure.

$đq$ — Heat input per unit mass of mixture.

q^+, q^- — Stresses applied to positive and negative sides of plate paralled to z axis.

Q_x, Q_y — Shear resultants per unit length along edge of plate.

r — Radial co-ordinate.

R — Pore compressibility constant.

$R(t)$ — Viscoelastic material time operator.

S_{ik} — Total stress tensor for interacting media.

s_1, s_2 — Interacting continua.

s — Laplace transform variable, or specific entropy per unit mass.

S — Displacement generating function.

t — Time variable.

Δt — Finite time increment.

T — Absolute temperature.

$\dot{u}_i$ — Velocity vector of solid particle in mixture.

u_r , u_z — Physical components of displacement vector in $\hat{e}_r$ and $\hat{e}_z$ directions.

u — In-plane plate displacement in x direction.

V — Volume of interacting mixture.

Vp — Pore volume.

$\dot{v}_i$ — Velocity vector of fluid particle in mixture.

V_l — Volume of solid material contained in mixture.

v — Lateral displacement in plane of plate in y direction.

w — Plate deflection in z direction.

$\bar{\bar{w}}_0$ — Iterated Laplace-Hankel zero order transform of plate deflection.

x_i — Position vector of solid particle in interacting mixture.

X_i — Initial co-ordinate position of solid particles prior to deformation.

y_i — Position vector fluid particle in interacting mixture.

Y_l — Initial co-ordinate position of fluid particle prior to deformation of interacting mixture.

z — Vertical co-ordinate.

Z — Function of Hankel variable, η.

LIST OF SYMBOLS

Greek Characters

$\alpha_1, \alpha_2, \alpha_4, \alpha_5, \alpha_6, \alpha_8$ – Materials constants.

γ – Fluid density function of interacting mixture.

δ – Unjacketed compressibility constant.

δ_{ij} – Kronecker delta.

ε_{ij} – Strain tensor in viscoelastic plate.

ε'_{ij} – Deviatoric strain tensor in viscoelastic plate.

ζ – Measured fluid outflow per unit volume.

η – Hankel transform variable.

κ – Jacketed compressibility constant.

λ_1, λ_2 – Constants.

μ – Variable.

ν – Poisson's ratio of plate material, or order of Bessel Function.

π_i – Diffusive force.

π_{ik} – Partial fluid stress tensor

$^{(\,)}\bar{\bar{\rho}}$ – Initial density of fluid or solid prior to mixing.

$^{(\,)}\bar{\rho}$ – Density of fluid or solid reckoned per unit volume of mixture prior to loading.

$^{(\,)}\rho$ – Density of fluid or solid reckoned per unit volume of mixture at any time, t.

σ_{ik} – Solid partial stress tensor. Also, viscoelastic material stress tensor.

σ'_{ik} – Deviatoric stress tensor for viscoelastic material.

τ – A dummy variable of integration.

ϕ – Stress relaxation function for viscoelastic material.

I

INTRODUCTION

This research was conducted in cooperation with the Federal Highway Administration under the Highway Planning and Research Program to assist in the national effort to develop a rational method for designing flexible highway pavements. The work undertaken in this study consists of the solution of a stress distribution theory which accounts for interaction of water and solid materials in the soil foundation, and the viscous time-dependent effect in the pavement material. The viscous effect is included in order to make the mathematical model of the pavement structure as realistic as feasible since the viscous time effect is very significant in bituminous concrete. However, the primary purpose of the research is to develop an analytical tool for investigating water-solid interaction in saturated pavement structures subjected to external loads.

The specific objectives, as outlined in the Project Proposal, were to:

1. Determine the displacement field which results when a linear viscoelastic plate supported on a porous elastic half-space containing an incompressible fluid is subject to quasi-static point loading normal to the half-space.

2. Determine the resulting stress field in the half-space in the situation discussed in Objective 1.

3. Provide a computer program in FORTRAN language which will be of use to other investigators in considering the stress field and displacement field

problems discussed in Objectives 1 and 2. This program would permit the users to vary pertinent parameters, and to solve various quasi-static type vertical loading problems.

The actual loading considered was a uniform circular load rather than a point load as indicated in Objective 1. The circular loading pattern was adopted because it approximates the load pattern of a single wheel load, and can be simulated experimentally by plate loading tests. This problem has not been treated before,either analytically or numerically.

The main product of the study is the solution of the viscoelastic plate on poro-elastic half-space problem. In conducting the research necessary to solve the defined problem, two secondary new results were obtained. The first of these was the determination of a new initial condition on the poro-elastic foundation that allows for compressibility in the elastic material which makes up the skeletal porous structure of the medium; the second consisted of the development of a numerical technique for inverting iterated Laplace-Hankel transform functions.

Before proceeding with the mathematical problem it might be appropriate to discuss the problem on purely physical grounds in order to illustrate the significance of the study and its relationship to the overall problem of analyzing flexible pavement structures.

To begin with, the net effect of the solid-fluid interaction in the soil foundation is that the hydrostatic stress that is applied to the soil particles is less than that applied to an element of the soil mass. The reason for this is that part of the load that is applied to the soil is carried by the fluid. At points in the foundation where the pressure on the fluid is large, the soil structure is weakened, because the ultimate undrained shear strength of the soil material decreases as the effective pressure on the solid particles decreases.

The solid-fluid interaction under load phenomenon is time-dependent and becomes insignificant after some period of time. The nature of the decay of this transient effect is of interest because as the interaction reaches a steady state condition, the effective solid particle stresses approach the applied stresses and the fluid pressure approaches zero value. The reason that the interaction dissipates is that the fluid in the soil foundation tends to move from areas of high total pressure to areas of low pressure. As the fluid moves out of a highly stressed region, the solid particles in the element are required to carry a greater portion of the load in order to maintain equilibrium.

The deformation of the foundation is time-dependent because, as the fluid flows away from highly stressed regions, the soil decreases in volume and settlement occurs. This means that if a stationary load is placed on the pavement structure for a period of time, the pavement will continue to deform for some time after the load is applied due to the solid-water interaction occurring in the foundation. The viscous properties of the bituminous concrete cause this effect to be magnified because, although the bituminous material exhibits a relatively high initial stiffness, it creeps under constant loading.

In actual service, loading that is applied to highway pavements is transient in nature. The loading that is applied at a point in the foundation increases as a vehicle approaches and decreases as it departs; and this process is repeated many times during the service life of the pavement. The net effect of the service loading can be approximated by utilizing superposition techniques. In this research, the effect of a uniform circular load--applied instantly and then held constant indefinitely--is studied. By adding up the effect of several such loads applied at different times, any smooth time-dependent loading can be approximated. Therefore, it is apparent that the solution determined in this study is a building block which can be used to approximate more complicated time-dependent pavement loadings if necessary.

The viscous property of the bituminous concrete is temperature sensitive. However, isothermal deformation is assumed in the subsequent analysis presented in this report. Therefore, in applying the analysis it is necessary to regard the deformation as occurring at some specified temperature. Considering the service loading that is to be ultimately approximated, it seems reasonable to assume that the temperature of the pavement will remain constant during the passage of one service load. The temperature distribution in the pavement slab can be expected to vary with depth; however, the bituminous layer thickness is relatively thin (less than five inches) in the pavement structures being considered in this study and, therefore, the temperature gradient can be expected to be small. An allowance for the temperature distribution with respect to depth can be made by selecting the assumed constant temperature value equal to the average of the actual distribution. In superimposing the effect of several service loads, each load may be assumed to occur at different specified temperatures. This can be done by utilizing appropriate material property data in analyzing the effect of each of the loads.

This report of the research accomplished is organized into eleven chapters. In Chapter II, a mathematical model of the pavement structure is introduced. The ideal poro-elastic material, which simulates the foundation in the model, is then discussed in Chapters III and IV. In Chapter V, the ideal linear viscoelastic material is discussed. This material is used to simulate the bituminous pavement layer in the model. Thin viscoelastic plate theory is introduced in Chapter VI. The mathematical solution of the model response to external load is given in Chapters VII, VIII, and IX. In Chapter X, the results of the analysis are applied to a hypothetical pavement structure using measured material property data. Numerical results are given that include fluid pressure distribution and decay curves. Finally, in Chapter XI, some conclusions and recommendations are presented which concern the application and extension of the results obtained in this study. The computer program which was developed to perform the pavement analysis is included in the Appendix. The program is designed so that other investigators may use it to perform stress analysis of flexible pavement structures. Laboratory tests for determining the material constants required are also discussed in the Appendix.

II

THE MATHEMATICAL MODEL

A mathematical model of the flexible pavement structure may be constructed using ideal materials to simulate the actual physical materials. Once this is accomplished, if the model is then assumed to be subjected to loading, it may be analyzed to obtain an approximation of the deformation that would occur if the real structure were subjected to the same loading.

The mathematical model that is used to simulate the flexible pavement structure in this study consists of a linear viscoelastic plate supported on a linear poro-elastic solid as shown in Figure 1. The plate is of infinite extent and the foundation occupies the half-space lying directly below the plate. A vertical step load is applied to the plate at zero time. The load is uniformly distributed over a circular area of radius b on the plate surface. The model is considered to have been at rest and unstressed prior to application of the load.

The following assumptions have been made concerning the physical response of the model:

1) body forces are neglected (A body force solution may be superimposed on the results obtained.)

2) the deformation occurs at a constant temperature

3) the deformation of the model is quasi-static (independent of inertial effects)

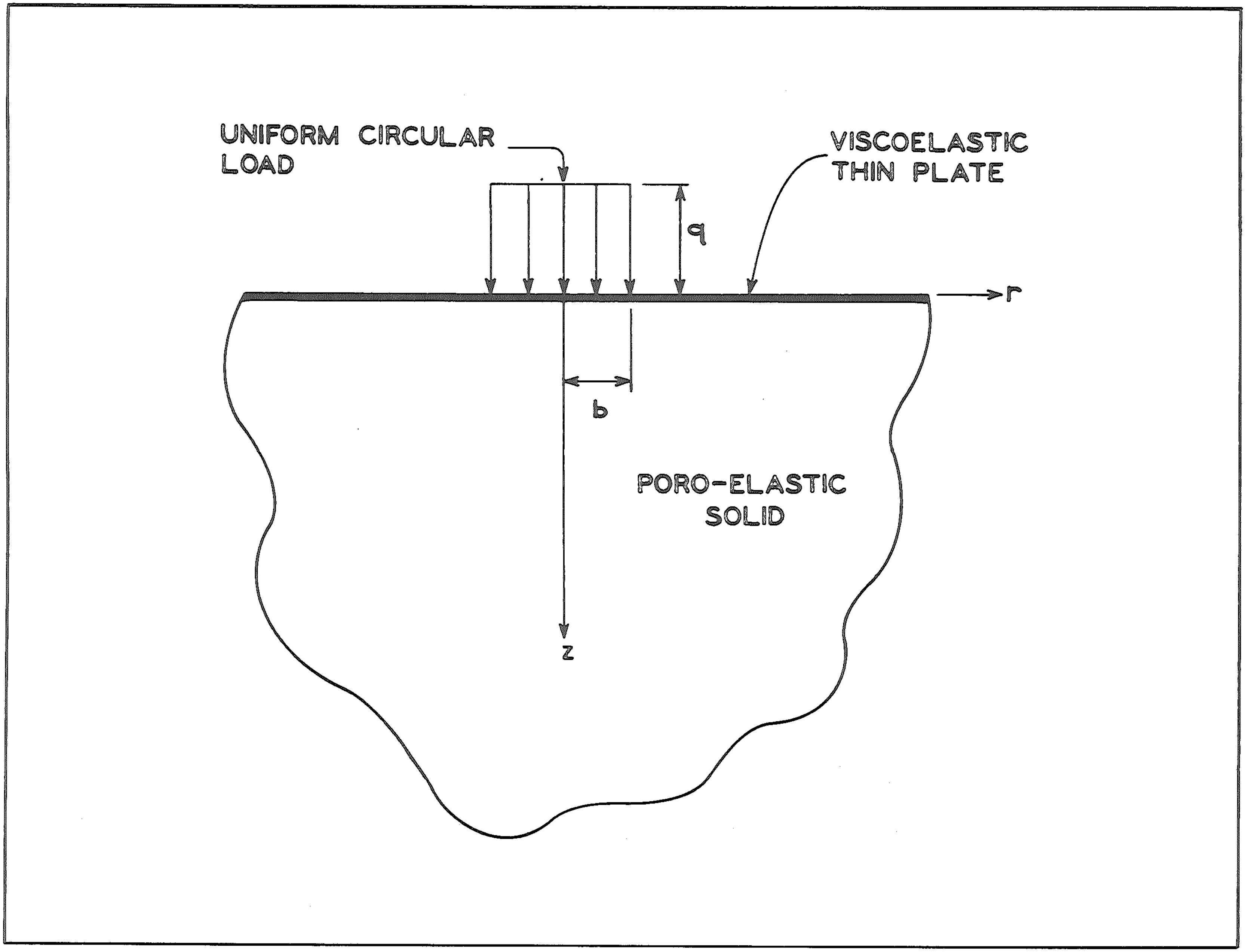

Figure 1. Mathematical model.

4) the model is assumed to be at rest prior to application of the load

5) the displacements and velocities at all points in the model are small so that linear theory is applicable

6) the viscoelastic plate material is assumed to be incompressible

7) the boundary between the plate and the foundation is frictionless and, therefore, only normal stresses are transferred from the plate to the solid

8) the vertical displacements of the plate are equal to that of the foundation at the interface

9) the thin viscoelastic plate forms an impermeable boundary across which no fluid can flow

10) the poro-elastic foundation consists of a linear, isotropic, perfectly elastic solid and an incompressible fluid

11) the poro-elastic foundation is homogeneous and has isotropic mechanical properties

12) creeping flow of the fluid in the foundation is assumed (Viscous effects on the flow through the pores are accounted for by introducing a permeability parameter.)

13) the shear stresses acting on the fluid are assumed to be small in comparison to the normal stresses and, therefore, may be neglected in the solution of stress distribution problems in the foundation.

The analysis of the mathematical model is summarized in the computer program that is presented in Appendix A. The program can be used to determine stresses and displacements throughout the model. The theoretical basis for the computer program is developed in the text that follows. A summary of the development is presented here for reference.

To begin with, it is necessary to define the mechanics of deformation for the interacting media foundation. This is done in Chapter III of the text, where the necessary field equations and constitutive equations are presented. This system of equations is reduced in Chapter IV by substituting the constitutive equations into the equilibrium equations. Further reduction is accomplished by considering the axial symmetry of the foundation loading. Two unknown displacement generating functions are introduced which define the axisymmetric deformation.

Next, in Chapter V, the mechanical properties of the viscoelastic plate are investigated. Constitutive equations are presented that define the deformation response to load for the viscoelastic material. Then, by utilizing the continuum equilibrium equations, the constitutive equations, and the geometry of the plate, a single differential equation defining the deformation response of the plate is obtained in Chapter VI. By applying the methods of operational calculus it is possible to reduce the plate equation to an algebraic expression involving iterated Laplace-Hankel transforms of the unknown plate deflection and foundation reaction.

In Chapter VII, an initial-boundary value problem is defined for the foundation. The unknown foundation reaction is one boundary condition of the problem. In Chapter VIII, the boundary value problem is solved to obtain an expression for the Laplace-Hankel transform of the foundation reaction. The expression that is obtained is combined with that determined in Chapter VI to yield explicit expressions for the transformed plate deflection and foundation reaction. The transformed solutions of other unknown stresses and displacements in the foundation are then determined using the deflection and foundation reaction expressions.

In order to determine physically meaningful solutions, it is necessary to invert the transformed solution images. This matter is dealt with in Chapter IX. The inverse Hankel transformation is approximated using a numerical integration algorithm. The inverse Laplace transformation is accomplished by approximating the time-dependent solution with a generalized Fourier series. Knowledge of the Laplace transform of the solution permits construction of the series approximation.

III

MECHANICS OF INTERACTING MEDIA

In order to solve the plate on half-space problem it is necessary to mathematically describe the plate and foundation so that the applied load can be related to the deformation. The interacting mixture of solid and fluid that makes up the foundation will be considered first.

Biot [1] formulated a theory of deforming porous media in 1939. His formulation was proposed to mathematically describe the porous solid model proposed by Terzahgi [2] to mechanically depict soil consolidation under load. The Biot theory has been developed and applied by many researchers since its conception and has become the classical theory of poro-elasticity. Recently, Paria [3] has discussed the Biot theory and its history of development.

The mechanics of interacting media treated here was formalized more recently (1965) by Green and Naghdi [4] and is based on thermodynamic considerations. The mechanics of interacting media are quite general and applicable to mixtures of gases, solids, and fluids. In the case of a mixture of a perfectly elastic solid and an incompressible fluid, the modern theory reduces to the classical formulation of poro-elasticity. In 1968, Tabaddor [5] discussed the development of the modern theory and its relationship with poro-elasticity and theories of flow through rigid media. In the present paper, the interacting media theory will be discussed for the case when the mixture forms a poro-elastic material.

3.1 Fundamental Properties

In the development of the mechanics of interacting media, certain basic assumptions have been made [4, 5] and will be referred to here as basic postulates.

> Postulate I: At every point in a poro-elastic continuum there exists both fluid and solid, neither of which may be isolated physically from the mixture.

Physically, Postulate I can be examined based on the microscopic physical model which has been proposed to describe poro-elastic material [1, 3]. Microscopically, the material is assumed to consist of an elastic skeleton containing many interconnected void spaces, or pores, which are filled with fluid. A small material element is thus seen to always contain both fluid and solid. However, when macroscopic structures of poro-elastic materials are viewed, the small element appears to be a point in the structure. (For example, in the problem being considered here the macroscopic structure is a half-space of infinite extent.) Therefore, it is assumed that the small element of material can be considered a point in the macroscopic structure, and in this sense there exists both fluid and solid at every point in the mixture.

The mixture of solid and fluid at an arbitrary point in the mixture will take on new density properties. The density properties of the mixture at a point in the macroscopic body can be examined by first considering a small finite element and then shrinking it to a point. The fluid in the element has a true density of ${}^{(2)}\bar{\bar{\rho}}$ which is equal to its mass divided by its mass volume. An artificial initial fluid density ${}^{(2)}\bar{\rho}$ can be defined by dividing the fluid mass in the poro-elastic element by the total volume of the element. This density ${}^{(2)}\bar{\rho}$ can be thought of as the apparent initial density of the fluid in the poro-elastic volume element. Further, if the element is deformed such that its volume changes, and possibly some of the fluid flows out of it, a new artificial density ${}^{(2)}\rho$ may be defined as the ratio of the current fluid mass in the element and the current volume of the element. ${}^{(2)}\rho$ can then be considered to be the apparent fluid density of the deformed poro-elastic material at time t. Similarly, the elastic solid in the skeletal structure of the volume element can be seen to have a real density ${}^{(1)}\bar{\bar{\rho}}$, an artificial density ${}^{(1)}\bar{\rho}$ at time zero, and a density ${}^{(1)}\rho$ at time t equal to its mass divided by the deformed volume of the element. Now, if the volume element is shrunk to a point, and it is assumed both fluid and solid exist at the point, it must be assumed that the densities defined above do not change.

In order to proceed with the development of the mechanics of interacting media, it is necessary to state Postulate I mathematically. Considering a mixture of two continua s_1 and s_2, that are in relative motion, the position of each point in s_1 and in s_2 are given by Eq. (1):

$$x_i = x_i(X_1, X_2, X_3, t), \quad y_i = y_i(Y_1, Y_2, Y_3, t), \quad i = 1,2,3 \qquad (1)$$

where x_i denotes the position of each point in s_1 and y_i that of s_2. Both position vectors x_i and y_i are expressed in Cartesian co-ordinates and are measured from a common Cartesian reference frame. The co-ordinates X_i and Y_i refer to the original positions of the points in s_1 and s_2 prior to deformation. Postulate I states that every point in the mixture is occupied simultaneously by a point in s_1 and one in s_2. Focusing on an arbitrary point in the mixture it is apparent that some point that was originally at X_1 in s_1, and one that was at Y_1 in s_2, now occupy the arbitrary point in the mixture as indicated by Eq. (2):

$$x_i = y_i \, , \; i = 1,2,3 \qquad (2)$$

Since Eq. (2) is true for any arbitrary point in the mixture, it is true throughout the mixture at any given time t. It can be seen, therefore, that Eq. (2) is mathematically equivalent to Postulate I.

> Postulate II: The total stresses that are applied to an infinitesimal volume element at a point in the poro-elastic body are equal to the sum of the partial stresses on the solid and fluid continua at that point.

Physically, this postulate can be understood by considering a small finite element of the poro-elastic mixture which is subjected to surface traction stresses on all sides. If the total force applied on any given face is divided into that applied to the solid and that applied to the fluid, then two partial stress components can be defined by dividing these two forces by the gross area of the element surface. Letting s_{ik} denote the total stress traction components applied to the cubic element of the poro-elastic mixture, it can be seen that partial solid and fluid stress components σ_{ik} and π_{ik} are related to the total stress components by Eq. (3).

$$\sigma_{ik} + \pi_{ik} = s_{ik} \qquad (3)$$

Now if the cube is shrunk to a point, or more accurately, if it is assumed to be infinitesimal relative to the poro-elastic body, then σ_{ik} and π_{ik} may be thought of as the stresses applied to the two artificial continua that exist at every point in the mixture according to Postulate I. The stresses on the two continua, therefore, must satisfy Eq. (3) which is equivalent to the condition stated in Postulate II.

Another basic property that is exhibited by the interacting media considered in this research is that of porosity. The poro-elastic mixture is assumed to have a strong skeletal structure as opposed to a colloidal mixture of solid and fluid having no pore structure. It is therefore possible to define the porosity f of an element of the skeletal structure as shown in Eq. (4):

$$f = \frac{V_p}{V} \tag{4}$$

where V_p denotes the volume of the interconnected pores, or void spaces, and V is the total volume of the element. An alternative definition of porosity may be derived by considering the mean pore area of the surfaces that are oriented parallel to one face of a cubic element of the skeletal structure. The mean pore area A_p is defined by Eq. (5):

$$A_p = \frac{1}{h}\int_0^h m(z)\,dz \tag{5}$$

where h denotes the length of the element and m the ratio of pore area to total area on the surfaces parallel to the z faces of the element. The volume V_p can be computed in terms of A_p as shown in Eq. (6).

$$V_p = A_p h \tag{6}$$

Substituting Eq. (6) into Eq. (4) yields an alternative expression of porosity in terms of A_p and the gross area of the element face A.

$$f = \frac{A_p}{A} \tag{7}$$

This second definition of porosity is not used in the subsequent development of the mechanics of poro-elasticity. It is used, however, in interpreting laboratory tests which are performed to determine the material constants.

3.2 Field Equations for Interacting Media

The approach used by Green and Naghdi [4] to establish field equations was to first assume that the interacting mixture satisfied Postulates I and II which were given in 3.1. They then considered the thermodynamics of the interacting mixture and adopted a third postulate.

> Postulate III: The first law of thermodynamics as stated in Eq. (8) applies to interacting media.

Rate of Increase of energy in a mass system.	=	Power of the external forces on the mass system.	-	Rate of flow of mechanical and non-mechanical energy into the mass system.	(8)

An energy balance equation which expressed Eq. (8) in mathematical terms was derived for a fixed arbitrary volume in space, and invariance conditions which result when the effect of rigid body motions of the mixture are considered were systematically applied. The equations of mass conservation and motion that were derived using this technique are analogous to the well known field equations for single constituent continua, such as for example, the elastic solid.

One of the expressions obtained, the conservation of mass equation, is given in Eq. (9):

$$\frac{D^{(1)}\rho}{Dt} + {}^{(1)}\rho\dot{u}_{k,k} + \frac{D^{(2)}\rho}{Dt} + {}^{(2)}\rho\dot{v}_{k,k} = 0 \qquad (9)$$

In Eq. (9), $\frac{D}{Dt}$ is the material time-derivative operator which measures the rate of change in density in this case of any given particle of material as observed from a fixed point in space. $\dot{u}_{k,k}$ and $\dot{v}_{k,k}$ are the divergences of the solid and fluid particle velocities measured with respect to the same spatial point. In the usual case, with no chemical action taking place between the fluid and solid components in the mixture, each mass element is conserved and stronger continuity conditions hold.

$$^{(1)}m \equiv \frac{D^{(1)}\rho}{Dt} + {}^{(1)}\rho \dot{u}_{k,k} = 0 \tag{10}$$

$$^{(2)}m \equiv \frac{D^{(2)}\rho}{Dt} + {}^{(2)}\rho \dot{v}_{k,k} = 0 \tag{11}$$

The equations of motion that were obtained are shown in Eq. (12).

$$(\sigma_{ki} + \pi_{ki})_{,k} + {}^{(1)}\rho F_i + {}^{(2)}\rho G_i = {}^{(1)}\rho f_i + {}^{(2)}\rho g_i + {}^{(1)}m \dot{u}_i + {}^{(2)}m \dot{v}_i \qquad (12)^{1}$$

The new variables F_i, G_i, f_i and g_i, that are introduced in Eq. (12) are, respectively, the body forces per unit mass on the solid and fluid continua and the acceleration vectors of the solid and fluid particles. The last two terms in Eq. (12) vanish according to the strong continuity conditions. For quasi-static deformations, the two acceleration terms on the right side of Eq. (12) may also be dropped. In the absence of body forces and with quasi-static deformation Eq. (12) reduces to Eq. (13) which is the form used in this study.

$$(\sigma_{ki} + \pi_{ki})_{,k} = 0 \tag{13}$$

A diffusive force vector π_i was introduced to combine certain force type terms that appear in the energy balance equation and are attributed to interaction occurring between the two deforming media. π_i is defined as shown in Eq. (14):

$$\pi_i \equiv \frac{1}{2}(\sigma_{ki} - \pi_{ki})_{,k} + \frac{1}{2}{}^{(1)}\rho(F_i - f_i) - \frac{1}{2}{}^{(2)}\rho(G_i - g_i) \tag{14}$$

In the absence of body forces and for quasi-static deformation Eq. (14) reduces to Eq. (15):

$$\pi_i = \frac{1}{2}(\sigma_{ki} - \pi_{ki})_{,k} \tag{15}$$

Applying an invariance under rigid rotation argument to the energy balance equation Green and Naghdi showed that the total stress tensor must

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(1) The commas appearing in Eq. (9) and (12) indicate partial differentiation with respect to x_k. Repeated indices indicate summation. These conventions apply throughout the text.

be symmetric as shown in Eq. (16).

$$\sigma_{ki} + \pi_{ki} = \sigma_{ik} + \pi_{ik} \tag{16}$$

The principle results of this summary that will be used in the subsequent poro-elastic analysis are expressions (13) and (16) and definition (15) which are the quasi-static equilibrium equations, the total stress tensor symmetry relation, and the diffusive force definition. To complete the formulation of poro-elasticity it is necessary to obtain constitutive relationships for the stresses and diffusive forces that will relate these variables to the deformation of the media. Before considering the formulation of constitutive equations, however, it is useful to consider what constraints are placed on the deformation of the interacting mixture by the incompressibility of the pore fluid.

3.3 Incompressibility Condition

The fluid in the poro-elastic material is assumed to be incompressible. However, this does not imply that the artificial fluid continuum with density ${}^{(2)}\rho$ is incompressible. Physically, this becomes apparent when a small finite volume element of the poro-elastic material is considered. In such an element, the fluid is contained in pores which are surrounded by an elastic skeletal structure. Suppose, for example, that the cube faces are sealed so that none of the fluid can escape. Then when the cube is compressed the elastic skeleton will decrease in volume and the fluid filled pores will become more closely spaced. Therefore, the volume of the element can be decreased without changing the actual fluid content in it, and the artificial density ${}^{(2)}\rho$ which is equal to the fluid content divided by the current volume of the cube can be changed without removing any of the fluid from the element.

Consider as another example a permeable cube, and assume that elastic material in the cube to be incompressible also. In this case, if the cube is compressed for a period of time the skeletal structure will change shape but its volume will remain the same. As the pressure is applied the fluid will flow out of the pores, allowing the pores to decrease in volume. Therefore, the artificial density $^{(1)}\rho$ will change because the original elastic material becomes contained in an element of decreased volume. Thus, it can be seen that incompressibility of the elastic skeletal material does not imply incompressibility of the artificial elastic continuum. Other density phenomena can be predicted for the artificial interacting continua; however, further examples will not be pursued here as the two examples given serve to motivate the analysis that follows. The derivation that is given here follows the work of Tabaddor [5].

Consider a small unstressed element of poro-elastic material with initial porosity f initial solid volume $\bar{V}_1$ and volume V_1 at time t. The volume $\bar{V}_1$ is defined as the volume formed by the extreme boundary surfaces of the mixture. Although the actual volume of the elastic material in the skeletal structure is less than that of the cube, the artificial elastic solid used for the continuum approximation has exactly the same volume as the element of poro-elastic material. Since the elastic strains are assumed to be small, the change in volume per unit cube of the artificial elastic material may be taken equal to e_{mm}, the sum of the normal strains. It is, therefore, possible to express V_1 in terms of $\bar{V}_1$ as shown in Eq. (17):

$$V_1 = \bar{V}_1 (1 + e_{mm}) \qquad \text{where} \quad e_{mm} \equiv u_{m,m} \tag{17}$$

It is reasonable to assume further that there exists a linear relationship between the element compressibility and volume change of the pores. If such a relationship is assumed, and after the initial dilatation of the skeletal material has taken place, v is defined as the current pore volume per unit volume, then the pore volume may be expressed as shown in Eq. (18):

$$v = \bar{V}_1 (R (e_{mm} - e_{mm}(o)) + f) \tag{18}$$

where $e_{mm}(o)$ denotes the initial value of e_{mm} at time zero when the loading is applied, and where R denotes the ratio of pore compressibility to total element compressibility. R depends on the geometry of the skeletal structure and the nature of the material filling the pores. As the sample is compressed, the pores will become smaller. This is pore compressibility. However, the elastic skeletal structure will be compressed also and, therefore, the total change in volume of the cubic sample will be greater than the change in pore volume. If the pores are filled with an incompressible fluid, the pore volume change will depend on the fluid flow from the element. Also, in this case, strong interaction occurs between the fluid and solid and the change in volume of the skeletal structure is dependent upon the flow of fluid from the element too.

If the skeletal structure material is incompressible, the pore compressibility equals the total compressibility and $R = 1$. Biot and others [1, 8] have treated the analysis of such materials. In the present research, material having a compressible skeletal structure, such that R will be less than one and greater than zero, is analyzed.

It is desirable for analytical purposes to consider R as constant.

Postulate IV states the assumed condition:

> Postulate IV: The ratio of pore compressibility to total compressibility of a small finite element of the porous, fluid filled solid is assumed to be constant for times greater than zero.

From Postulate IV it is seen that the validity of the analysis will vary between different porous media according to how nearly R approaches a constant value for $t>0$ and for how long it remains so. Therefore, in investigating specific materials, this limitation should be measured by experimentally observing the behavior of R .

Treating R as constant, we proceed to determine the constraint implied by the pore fluid incompressibility. The total mass of fluid in the pores per unit volume at time t can be expressed as shown in Eq. (19):

$$
\begin{aligned}
{}^{(2)}\bar{\bar{\rho}}\, v &= {}^{(2)}\rho\, V_l \\
\frac{{}^{(2)}\rho}{{}^{(2)}\bar{\bar{\rho}}} &= \frac{v}{V_l}
\end{aligned}
\qquad (19)
$$

The second equality follows from the given expression. Similarly, the initial apparent density of the artificial fluid can be expressed by the first Eq. (20) which implies the second equality:

$$
\begin{aligned}
{}^{(2)}\bar{\rho} &= {}^{(2)}\bar{\bar{\rho}}\, f \\
{}^{(2)}\bar{\bar{\rho}} &= \frac{{}^{(2)}\bar{\rho}}{f}
\end{aligned}
\qquad (20)
$$

The ratio of current pore volume to current element volume can be expressed in terms of fluid densities, or in terms of apparent solid strain as shown in Eq. (21):[a]

$$\frac{f\,{}^{(2)}\rho}{{}^{(2)}\bar{\rho}} = \frac{v}{V_l} \simeq (e_{mm} - e_{mm}(0))(R-f) + f - f\,e_{mm}(0) \qquad (21)$$

---

[a] The final approximate equality was obtained from Eqs. (17) and (18) by performing synthetic division, and neglecting the second order terms in the result.

Equation (22) follows by equating the left and the right sides of Eq. (21), and rearranging terms.

$$\frac{f\,{}^{(2)}\rho}{{}^{(2)}\bar{\rho}} = e_{mm}(R-f) + f - R\,(e_{mm}(o)) \tag{22}$$

It is assumed that the deformation is isothermal, and that no chemical reaction occurs in the mixture. The last assumption implies that the mass elements of each component of the mixture are conserved during deformation. According to these assumptions the current apparent fluid density ${}^{(2)}\rho$ can be linearly related to the apparent density at time $t = 0$ by a small scaler function γ as shown in Eq. (23):

$${}^{(2)}\rho = {}^{(2)}\rho(o) + \gamma \tag{23}$$

Substituting Eq. (23) into the conservation of fluid mass equation (Eq. (11) of 3.2) and neglecting the second order terms, results in Eq. (24):

$$\frac{\partial \gamma}{\partial t} = -{}^{(2)}\rho(o)\frac{\partial v_{k,k}}{\partial t} = -\frac{{}^{(2)}\bar{\rho}}{(1+e_{mm}(o))}\,\frac{\partial v_{k,k}}{\partial t} \tag{24}$$

If the partial time derivative is taken in Eq. (22), Eq. (25) results.

$$\frac{f}{{}^{(2)}\bar{\rho}}\,\frac{\partial \gamma}{\partial t} = (R-f)\frac{\partial e_{mm}}{\partial t} \tag{25}$$

Substituting Eq. (24) in Eq. (25) results in the desired constraint condition Eq. (26).

$$\frac{\partial v_{k,k}}{\partial t} = -\frac{(R-f)}{f}\,\frac{\partial e_{mm}}{\partial t}\,(1+e_{mm}(o)) \tag{26}$$

or [3]

$$\frac{\partial v_{k,k}}{\partial t} \simeq \frac{(f-R)}{f}\,\frac{\partial e_{mm}}{\partial t}$$

This equation states the condition that is forced on $\frac{\partial v_{k,k}}{\partial t}$ and $\frac{\partial e_{mm}}{\partial t}$ by assuming that the pore fluid is incompressible.

3.4 Constitutive Equations

Field equations that insure equilibrium and continuity of mass throughout the interacting mixture have been given in 3.2. It is also necessary,

[3] The approximate equality on the left of the second Eq. (26) follows from the assumption that the first invariant of the solid strains is small, that is, much less than unity.

however, to relate the partial stress tensors at any arbitrary point in the mixture to the deformation at that point. When this has been accomplished it will be possible to relate deformations occurring in the interacting media to stresses applied at the boundary of the media in terms of a boundary value problem.

Tabaddor [5] has derived constitutive equations for a poro-elastic mixture consisting of a perfectly elastic solid and an incompressible viscous fluid for the case of small strain and strain rates. His work follows the more general work of Green, Naghdi, and Steel [4, 6]. The following fifth postulate led to the results obtained:

> Postulate V. The mixture of interacting media satisfies the second law of thermodynamics.

The second law of thermodynamics postulates the existence of an entropy state function s which satisfies Eq. (27):

$$\Delta s \geqslant \int_{state\ 1}^{state\ 2} \left(\frac{đq}{T}\right) \tag{27}$$

where Δs = change in specific entropy per unit mass

T = absolute temperature

$đq$ = heat input per unit mass which is not an exact differential

The equality sign in Eq. (27) holds for reversible processes, and the inequality sign for irreversible processes. The change in entropy is greater than that produced by heat input in the irreversible case because internal entropy production occurs due to dissipative processes, such as internal friction.

Green and Naghdi [4] constructed an entropy production inequality for a system of interacting media undergoing a reversible or irreversible process. This was done by examining entropy production of the mass system instantly occupying an arbitrary volume V in space. Since the expression obtained applies for any arbitrary choice of volume, the integral form of inequality may be localized to an arbitrary point. Green and Steel [6] expressed the condition at a point, mathematically, in terms of a localized entropy production inequality. In order to make use of the inequality it is necessary to construct the general form of the constitutive equations.

Green and Steel, in considering the problem of postulating constitutive equations for a mixture consisting of a non-linear elastic solid and viscous Newtonean fluid, relied on the known constitutive properties of each component for inspiration. The condition of isotropy was imposed on the assumed equations. The assumed constitutive relationships were then substituted into the entropy production inequality. This resulted in an inequality which contained several unknown state variables having undetermined coefficient functions. By arbitrarily varying the state variables one at a time, constraints on the unknown coefficients were obtained. These conditions were then substituted back into the assumed constitutive relationships. In this manner, Green and Steel succeeded in obtaining constitutive equations for the non-linear elastic and viscous fluid mixture. They then linearized the theory for the case of small elastic strains.

Tabaddor [5] investigated the same mixture with one additional physical restriction. He assumed the fluid to be incompressible. He derived a constraint on the rate of deformation tensors as a consequence of the fluid incompressibility. The incompressibility condition was presented in 3. 3. Tabaddor substituted his incompressibility Eq. (28) in the entropy production inequality.

$$\frac{\partial v_{m,m}}{\partial t} + a_1 \frac{\partial u_{m,m}}{\partial t} = 0 \qquad (28)$$

He then applied the procedure discussed above to determine the constraints placed on the unknown coefficients of the state variables in the entropy inequality. This resulted in a set of constitutive equations for a non-linear elastic solid and a viscous incompressible fluid. Tabaddor simplified his constitutive equations further by introducing the infinitestimal strain assumption. The resulting constitutive equations are:

$$\begin{aligned}
\sigma_{ik} = \sigma_{ki} &= a_1 \delta_{ik} + (a_4 - a_1 \frac{^{(1)}\bar{\rho}}{\bar{\rho}}) e_{mm} \delta_{ik} + 2(a_1 + a_2) e_{ik} \\
&+ (a_8 + \frac{a_1}{\bar{\rho}}) \gamma \delta_{ik} + a_1 \bar{p} \delta_{ik} \\
\pi_{ik} = \pi_{ki} &= -\Big({}^{(2)}\bar{\rho}\, a_8 - \bar{p} + \Big[{}^{(2)}\bar{\rho}\, a_6 + ({}^{(2)}\bar{\rho} + \bar{\rho}) \frac{a_2}{\bar{\rho}} \Big] \gamma \\
&+ {}^{(2)}\bar{\rho} (a_2 - \frac{^{(1)}\bar{\rho}}{\bar{\rho}} a_2) e_{mm} \Big) \delta_{ik} + \lambda_1 f_{rr} \delta_{ik} + 2\lambda_2 f_{ik} \\
\pi_i &= a(\frac{\partial u_i}{\partial t} - \frac{\partial v_i}{\partial t})
\end{aligned} \qquad (29)$$

where

$$f_{ik} \equiv \frac{1}{2}(\frac{\partial v_{i,k}}{\partial t} + \frac{\partial v_{k,i}}{\partial t})$$

and

$$e_{ik} \equiv \frac{1}{2}(u_{i,k} + u_{k,i})$$

If it is assumed that there are no initial stresses on the continua prior to application of the load at time zero, then Eq. (29) reduces to the following:

$$\begin{aligned}
\sigma_{ik} = \sigma_{ki} &= (a_4 e_{mm} + a_8 \gamma) \delta_{ik} + 2 a_2 e_{ik} \\
\pi_{ik} &= -({}^{(2)}\bar{\rho}\, a_6 \gamma + {}^{(2)}\bar{\rho}\, a_8 e_{mm}) \delta_{ik} + \lambda_1 f_{rr} \delta_{ik} + 2\lambda_2 f_{ik} \\
\pi_i &= a(\frac{\partial u_i}{\partial t} - \frac{\partial v_i}{\partial t})
\end{aligned} \qquad (30)$$

In order to reduce the constitutive equations of the fluid, it is assumed that the viscous terms, λ, f_{rr} and $2\lambda_2 f_{ik}$ are small in comparison to the hydrostatic pressure. Neglecting these terms in π_{ik} yields Eq. (31):

$$p = ({}^{(2)}\bar{\rho}\, a_6\, \gamma + {}^{(2)}\bar{\rho}\, a_8\, e_{mm}) \tag{31}$$

where π_{ik} has been replaced by $-p\,\delta_{ik}$ because the normal components of π_{ik} are equal and the shear components vanish.

A change of variable can be accomplished by considering Eq. (24) of 3.3 which reduces to Eq. (32):

$$\frac{\partial \gamma}{\partial t} = -{}^{(2)}\bar{\rho}\, \frac{\partial v_{k,k}}{\partial t} \tag{32}$$

because e_{mm} is assumed to be small.

Integrating this expression and noting that $\gamma = 0$ at $t = 0$ yields Eq. (33).

$$\gamma = -{}^{(2)}\bar{\rho}\, v_{k,k} \tag{33}$$

Substituting Eq. (32) for γ in Eqs. (30) and (31) yields Eq. (34).

$$\begin{aligned} \sigma_{ik} &= (a_4\, e_{mm} - a_8\, {}^{(2)}\bar{\rho}\, v_{m,m})\, \delta_{ik} + 2a_2\, e_{ik} \\ p &= -(a_6\, ({}^{(2)}\bar{\rho})^2\, v_{m,m} - a_8\, {}^{(2)}\bar{\rho}\, e_{mm}) \end{aligned} \tag{34}$$

The constitutive equations (34) are same as those given by Biot and Willis in equation(1) of reference [7].

The variable $v_{m,m}$ can be eliminated from the first constitutive equation (34) by solving for $v_{m,m}$ in the second equation and then treating p as unknown.

$$\sigma_{ik} = (a_4 - \frac{(a_8)^2}{a_6})\, e_{mm}\, \delta_{ik} + \left(\frac{a_8}{a_6\, {}^{(2)}\bar{\rho}}\right) p\, \delta_{ik} + 2\, a_2\, e_{ik} \tag{35}$$

Defining a_3 as the coefficient of e_{mm} and $-a_1$ as the coefficient of p in Eq. (35) results in the form of constitutive equations that are used in this study.

$$\begin{aligned} \sigma_{ik} &= -a_1\, p\, \delta_{ik} + 2\, a_2\, e_{ik} + a_3\, e_{mm}\, \delta_{ik} \\ \pi_{ik} &= -p\, \delta_{ik} \\ \pi_i &= a(\frac{\partial u_i}{\partial t} - \frac{\partial v_i}{\partial t}) \end{aligned} \tag{36}$$

The constitutive Eq. (36) relate the deformation occurring at any arbitrary point in the interacting media foundation to the state of stress there at times greater than zero. Since γ is zero initially at $t = 0$, the constitutive equations for σ_{ik} and p reduce to the following at the instant of loading.

$$\begin{aligned} \sigma_{ik} &= a_4 e_{mm} \delta_{ik} + 2a_2 e_{ik} \\ p &= a_8 {}^{(2)}\bar{\rho}\, e_{mm} \delta_{ik} \end{aligned} \tag{37}$$

IV

GOVERNING DIFFERENTIAL EQUATIONS OF PORO-ELASTICITY

4.1 The Differential Equations of Poro-Elasticity

Equations (1) through (5) summarize the results obtained in Chapter III:

$$(\sigma_{ki} + \pi_{ki}),_k = 0 \tag{1}$$

$$\sigma_{ki} = \sigma_{ik} = -a_1 p \delta_{ik} + 2a_2 e_{ik} + a_3 e_{mm} \delta_{ik} \tag{2}$$

$$\pi_{ki} = \pi_{ik} = -p \delta_{ik} \tag{3}$$

$$\pi_i = a\left(\frac{\partial u_i}{\partial t} - \frac{\partial v_i}{\partial t}\right) \tag{4}$$

$$f_{kk} = \frac{(f-R)}{f} \frac{\partial e_{mm}}{\partial t} \tag{5}$$

These equations govern the quasi-static deformation of an interacting continua mixture consisting of a linear elastic solid and an incompressible fluid at times greater than zero. Equations (1) are the equilibrium equations of the continua mixture. These expressions were obtained from the equations of motion, Eq. (12) of 3.2, by neglecting the acceleration terms (quasi-static assumption) and setting the body forces equal to zero. Equation (2) is the constitutive equation for the solid partial stress tensor σ_{ij} . Equation (3) defines the partial fluid stress tensor. Equation (4) gives a constitutive relationship for the diffusive resistance in the media π_i . (Diffusive resistance results because interaction occurs between the two component materials when the continua are deformed.) Finally, Eq. (5) is a constraint that is imposed on the system of equations because the fluid that fills the pores is incompressible.

The system of Eqs. (1) through (5) contains 29 unknowns and consists of 19 equations. However, if the six solid strain definitions, the three diffusive force definitions, and the definitions of the three normal components of deviator tensor f_{ij} are introduced, the system is complete; containing 31 unknowns and 31 equations. The purpose of this section is to reduce this system of equations to a four-by-four system of equations with the solid displacements u_i and the fluid pressure p being the unknowns.

$$\sigma_{ik} + \pi_{ik} = a_1 p \, \delta_{ik} - p \, \delta_{ik} + 2a_2 \, e_{ik} + a_3 e_{mm} \, \delta_{ik} \tag{6}$$

By definition, π_i is expressed as follows [4]:

$$\pi_i = \frac{1}{2}(\sigma_{ki} - \pi_{ki})_{,k} \tag{7}$$

if the body forces are zero and the deformation is quasi-static. Substituting this expression in Eq. (4) yields Eq. (8).

$$\frac{1}{2}(\sigma_{ki} - \pi_{ki})_{,k} = a\left(\frac{\partial u_i}{\partial t} - \frac{\partial v_i}{\partial t}\right) \tag{8}$$

If the equilibrium equation (1) is divided by two and subtracted from Eq. (8) the following expression results:

$$-\pi_{ki,k} = a\left(\frac{\partial u_i}{\partial t} - \frac{\partial v_i}{\partial t}\right) \tag{9}$$

The substitution of expression (3) for π_{ki} in Eq. (9) yields Eq. (10).

$$p_{,i} = a\left(\frac{\partial u_i}{\partial t} - \frac{\partial v_i}{\partial t}\right) \tag{10}$$

This expression is of the same form as the modified Darcy law proposed by Biot [1] to describe the creeping flow of a fluid through a deforming porous media.[4]

---

[4] The microscopic flow through the pores is viscous and temperature dependent. The viscous effects are accounted for in coefficient of Eq. (9), which relates the macroscopic fluid discharge through the media to the macroscopic pressure gradient $p_{,i}$. In order to account for the temperature effect, it would be necessary to consider the coefficient to be temperature dependent. In the present problem, however, a may be considered constant because the deformation is assumed to occur at a constant temperature.

Substituting constitutive equations (6) into the equilibrium equations (1) yields Eq. (11):

$$2a_2 e_{ik,k} + a_3 e_{mm,i} - (1 + a_1) p_{,i} = 0 \tag{11}$$

The artificial solid strain tensor is defined as shown in Eq. (12):

$$e_{ik} = \frac{1}{2}(u_{i,k} + u_{k,i}) \tag{12}$$

Taking the divergence of the solid strain tensor Eq. (12) gives Eq. (13):

$$e_{ik,k} = \frac{1}{2}(u_{i,kk} + u_{k,ik}) \tag{13}$$

Substituting Eq. (13) in Eq. (11) yields Eq. (14):

$$a_2 u_{i,kk} + (a_2 + a_3) e_{mm,i} - (a_1 + 1) p_{,i} = 0 \tag{14}$$

Using vector notation Eq. (14) can be expressed as shown in Eq. (15):

$$a_2 \nabla^2 \hat{u} + (a_2 + a_3) \hat{\nabla}(\hat{\nabla} \cdot \hat{u}) - (a_1 + 1) \hat{\nabla} p = 0 \tag{15}$$

Equation (15) is analogous to the Navier displacement vector equation of elasticity. Because of the presence of four unknowns u_i and p in the three Eqs. (14), another equation is necessary to provide a complete system. A fourth equation can be obtained taking the divergence of both sides of Eq. (10).

$$\nabla^2 p = a\left(\frac{\partial e_{mm}}{\partial t} - f_{mm}\right) \tag{16}$$

f_{mm} can be eliminated from Eq. (16) by substituting expression (5). Making such a substitution results in Eq. (17):

$$\nabla^2 p = \frac{aR}{f} \frac{\partial e_{mm}}{\partial t} \tag{17}$$

In order to eliminate the dependent variable p from Eq. (17), another expression for $\nabla^2 p$ can be obtained from Eq. (15) by taking the divergence of that equation.

$$\nabla^2 p = \frac{(2a_2 + a_3)}{(a_1 + 1)} \nabla^2 e_{mm} \tag{18}$$

Substituting expression (18) into Eq. (17) eliminates p as shown in Eq. (19).

$$\nabla^2 e_{mm} = \frac{a(a_1 + 1)}{(2a_2 + a_3)} \left(\frac{R}{f}\right) \frac{\partial e_{mm}}{\partial t} \equiv \frac{1}{c} \frac{\partial e_{mm}}{\partial t} \tag{19}$$

Equations (15) and (19) are the fundamental equations that govern the deformation of a porous linear elastic solid saturated with an incompressible fluid. The solution of Eqs. (15) and (19) in conjunction with sufficient boundary conditions will provide the solutions of the quasi-static problems of poro-elasticity.[6] In the next section, the solution of Eqs. (15) and (19) through the use of displacement generating functions is discussed for the case of axially symmetric deformation.

4.2 Axially Symmetric Deformation - Displacement Generating Functions

The deformation of porous media problems that will be considered in this study will be limited to the special case where the displacements are axially symmetric. In this case, it is possible to further simplify the equations of poro-elasticity. In this section, two unknown functions $E(r,z,t)$ and $S(r,z,t)$ will be introduced from which the displacements in the poro-elastic media can be computed. In order that the equations of poro-elasticity be satisfied, it is necessary that the functions introduced satisfy certain differential equations; however, the equations that must be satisfied are much simpler than those that govern u_i and p.

Equation (20) expresses the axisymmetric constraint on the solid displacement vector.

$$\vec{u} = u_r(r,z,t)\,\hat{e}_r + u_z(r,z,t)\,\hat{e}_z \tag{20}$$

The Laplace differential operators $\vec{\nabla}$ and ∇^2 which appear in the fundamental equations (15) and (19) have the form of Eq. (21).

$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}\,, \quad \vec{\nabla} \equiv \hat{e}_r\frac{\partial}{\partial r} + \hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial \theta} + e_z\frac{\partial}{\partial z} \tag{21}$$

in cylindrical co-ordinates where $\hat{e}_r$, $\hat{e}_\theta$ and $\hat{e}_z$ are unit vectors tangent to the co-ordinate curves.

---

6 Alternatively, if e_{mm} is considered as a fifth unknown, in addition to u_i and p, then Eqs. (15), (18), and (19) may be considered the basic governing equations.

Assuming a displacement field defined in terms of two unknown functions, E and S as shown in Eq. (22)

$$\hat{u}(r,z,t)=\hat{\nabla}(E(r,z,t)+zS(r,z,t))-2S(r,z,t)\hat{e}_z \tag{22}$$

where

$$\nabla^2 S(r,z,t)=0 \tag{23}$$

leads to solutions of Eqs. (15) and (19).

Taking the divergence of both sides of Eq. (22) shows that the dilatation that results from assumed displacement field depends on E only.

$$\hat{\nabla}\cdot\hat{u}=e_{mm}=\nabla^2 E \tag{24}$$

Substituting Eq. (24) into Eq. (19) of 4.1 yields the partial differential equation which E must satisfy.

$$\nabla^4 E=\frac{1}{c}\frac{\partial}{\partial t}\nabla^2 E \tag{25}$$ [8]

If Eq. (22) is substituted into Eq. (15) of 4.1, which must be satisfied also, an expression for $\hat{\nabla}p$ results.

$$\hat{\nabla}p=\hat{\nabla}\left(\frac{(2a_2+a_3)}{(1+a_1)}\nabla^2 E+\frac{2a_2}{(1+a_1)}\frac{\partial S}{\partial z}\right) \tag{26}$$

Solving partial differential equation (26) yields Eq. (27) for p :

$$p=\frac{(a_3+2a_2)}{(1+a_1)}\nabla^2 E+\frac{2a_2}{(1+a_1)}\frac{\partial S}{\partial z}+p_0(t) \tag{27}$$

where $p_0(t)$ is arbitrary. The physics of the problem being studied here insures that $p_0(t)\equiv 0$. That is, the initial fluid pressure at time $t=0^-$ is assumed equal to zero, and no uniform hydrostatic pressure is applied to the fluid continuum at any time $t\geq 0$. Therefore, Eq. (27) reduces to Eq. (28).

$$p=\frac{(a_3+2a_2)}{(1+a_1)}\nabla^2 E+\frac{2a_2}{(1+a_1)}\frac{\partial S}{\partial z} \tag{28}$$

---

[8] In considering the axisymmetric deformation of materials with pore compressibility R equal to unity, McNamee and Gibson [8] proposed two displacement generating functions that satisfied equations of the form of Eqs. (23) and (25).

The displacement field defined by Eq. (22) may be expressed in physical component form as shown in Eq. (29).

$$u_r = \frac{\partial E}{\partial r} + z\frac{\partial S}{\partial r}, \quad u_z = \frac{\partial E}{\partial z} + z\frac{\partial S}{\partial z} - S \tag{29}$$

Partial differential equations (23) and (25) are the equations that define E and S and thus the axisymmetric deformation of the poro-elastic foundation. In order to establish sufficient boundary conditions on these equations, it is necessary to consider the deformation that occurs in the plate that is supported by the foundation. Therefore, the equation that governs the plate deformation must now be considered. First, however, a few comments concerning the arbitrariness of E and S are given in 4.3.

4.3 Arbitrariness of E and S Functions

Functions E and S are arbitrary to the extent that the partial stress tensors are not affected if an arbitrary linear function is added to E or if a constant plus a term linear in r is added to S. If E is modified as shown in Eq. (30)

$$E' = E + b_1 + b_2 r + b_3 z \tag{30}$$

then rigid body displacements result as shown in Eq. (31).

$$u'_r = u_r + b_2, \quad u'_z = u_z + b_3 \tag{31}$$

It is observed that the addition of the constant b_1 does not affect the displacements. If S is modified by adding a constant b_4 as shown in Eq. (32),

$$S' = S + b_4 \tag{32}$$

then the rigid body displacement (33) results.

$$u'_z = u_z + b_4 \tag{33}$$

Further, if a linear r term is added to S as shown in Eq. (34)

$$S' = S + b_5 r, \quad \frac{1}{2}\left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right) = 2b_5 \tag{34}$$

a rigid body rotation results.

It is interesting to note that the addition to S of a term linear in z does not affect the artificial solid displacements, but does cause a change in the partial stress components on the artificial fluid continuum as shown in Eq. (35).

$$S' = S + b_5 z\,,\; u'_r = u_r\,,\; u'_z = u_z$$
$$-\pi'_{rr} = -\pi'_{\theta\theta} = -\pi'_{zz} = p' = p + \frac{2a_2 b_5}{(1+a_1)} \tag{35}$$

V

THE VISCOELASTIC MATERIAL

In order to define the deformation response to load in the plate, it is necessary to relate the deformations occurring at any arbitrary point in the plate to the state of stress at that point. It is the purpose of this Chapter to set down such constitutive equations for the viscoelastic plate material.

The constitutive equations of the ideal, linear, isotropic, viscoelastic solid exhibit a time-dependent relationship between stress and deformation. However, since the stresses must be linearly related to the strains at all times in the viscoelastic material, the constitutive equations are similar to those of ideal elasticity. A convenient form of the viscoelastic constitutive equations results if the dilatation is related to the first invariant of the stress tensor and the deviatoric strain is related to the deviatoric stress tensor as shown in Eq. (1).

$$\sigma_{mm} = \frac{R(t)}{3(1-2\nu(t))}\,\varepsilon_{mm}$$

$$\sigma'_{ij} = \frac{R(t)}{(1+\nu(t))}\,\varepsilon'_{ij} \qquad (1)$$

where

$$\sigma'_{ij} \equiv \sigma_{ij} - 1/3\,\sigma_{mm}\,\delta_{ij}$$

$$\varepsilon'_{ij} \equiv \varepsilon_{ij} - 1/3\,\varepsilon_{mm}\,\delta_{ij}$$

$R(t)$ and $\nu(t)$ are linear time-dependent operators which are analogeous to the Youngs modulus and Poisson's ratio constants that appear in constitutive equations of ideal elasticity. In this study it is assumed that the viscoelastic plate material is incompressible and, therefore, the operator $\nu(t)$ must be constant and equal to 1/2 as shown in Eq. (2).

$$\varepsilon_{mm} = \frac{3(1-2(\nu))}{R(t)} \sigma_{mm} = 0 \qquad (2)$$

Taking this constraint on $\nu(t)$ into account eliminates the first constitutive equation (1) and simplifies the others to the form (3).

$$\sigma'_{ij} = 2/3\, R(t)\, \varepsilon'_{ij} \qquad (3)$$

This expression shows that one linear time operator defines the constitutive behavior, if the viscoelastic material is incompressible. Using the deviatoric tensor definitions (1), it is possible to express Eq. (3) in terms of the ordinary stress and small strain tensorial components.

$$\varepsilon_{ij} = \frac{3}{2R(t)} \sigma_{ij} - \frac{1}{2R(t)} \sigma_{mm} \delta_{ij} \qquad (4)$$

In the case of uniaxial stress, Eq. (4) reduces to Eq. (5):

$$\sigma_{(jj)} = R(t)\, \varepsilon_{(jj)} \qquad \text{(no sum on } j\text{)} \qquad (5)$$

which shows that a uniaxial experiment can be used to derive $R(t)$.

A hereditary integral type constitutive equation is used to describe the viscoelastic plate material in uniaxial strain for this study. This form of equation can be constructed if the stress relaxation behavior of the material is known. The necessary data are obtained by subjecting the material to a constant uniaxial strain and observing the stress that results as a function of time. The stress relaxation function $\phi(t)$ is defined in terms of the uniaxial test data as shown in Eq. (6):

$$\phi(t) = \frac{\sigma_{(jj)}(t)}{\varepsilon_{(jj)}(0)} \qquad \text{(no sum on } j\text{)} \qquad (6)$$

where the instant of load application is defined as time zero.

The general form of a typical $\phi(t)$ function is shown in Figure 2.

In order to approximate the stress response for a more complicated strain loading history, the linearity of the material can be utilized. This is done by superimposing the stress relaxation response to several step function loads such as shown in Figure 3, where the actual loading curve $\epsilon_{(jj)}(t)$ is approximated by either of the two step function curves shown. The stress response to each of the approximate loading curves is given by Eq. (7).

$$\sigma_u = \sum_k \phi_u(t-t_k)\frac{\Delta\epsilon_u}{\Delta t_k}\Delta t_k \simeq \sigma_{(jj)}(t)$$
$$\sigma_\ell = \sum_k \phi_\ell(t-t_k)\frac{\Delta\epsilon_\ell}{\Delta t_k}\Delta t_k \simeq \sigma_{(jj)}(t) \tag{7}$$

Letting Δt_k approach zero in both summations σ_u and σ_ℓ yields Eq. (8):

$$\textit{infinum}\ \sigma_u = \textit{supremum}\ \sigma_\ell = \int_{t_0}^{t} \phi(t-\tau)\frac{d\epsilon_{jj}(t)}{d\tau}\,d\tau \tag{8}$$

where the last equality follows from the definition of the Riemann Integral.

Letting t_0 approach $-\infty$ yields an expression for $\sigma(t)$ which accounts for the effect of the entire uniaxial strain history of material. Integrating the resulting expression by parts and assuming that the material was unstrained at $t = -\infty$ gives Eq. (9).

$$\int_{-\infty}^{t} \phi(t-\tau)\frac{d\epsilon_{(jj)}(\tau)}{d\tau}\,dt = \phi(0)\,\epsilon_{(jj)}(t) + \int_{-\infty}^{t} \epsilon_{(jj)}(\tau)\frac{\partial\phi(t-\tau)}{\partial\tau}\,d\tau \tag{9}$$

For the problem being considered it is assumed the loading history of the material prior to application of the strain loading at time zero may be neglected. This assumption leads to the hereditary integral constitutive equation (10) for uniaxial strain.

$$\sigma_{(jj)}(t) = \phi(o)\,\epsilon_{(jj)}(t) + \int_{0}^{t} \epsilon_{(jj)}(t)\,\frac{\partial\phi(t-\tau)}{\partial\tau}\,d\tau \tag{10}$$

Eq. (10) is equivalent to the operator form (11).

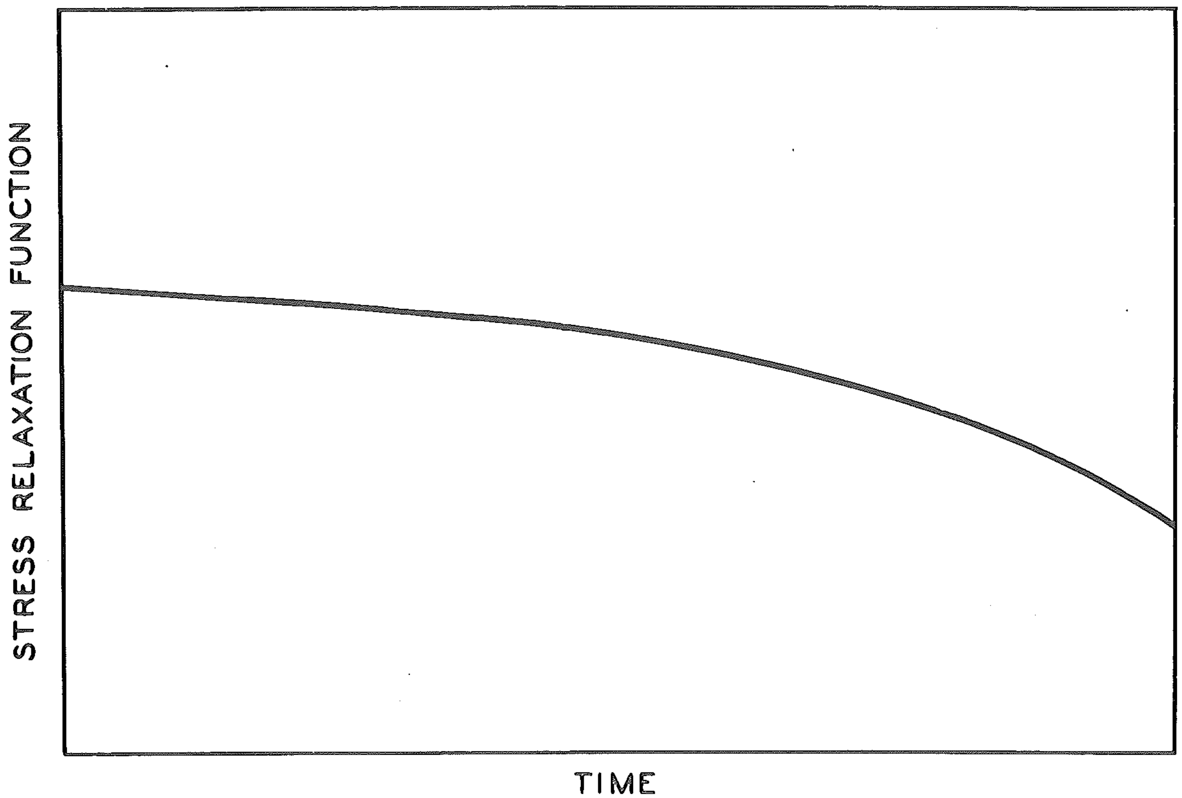


Figure 2. Typical stress relaxation function.

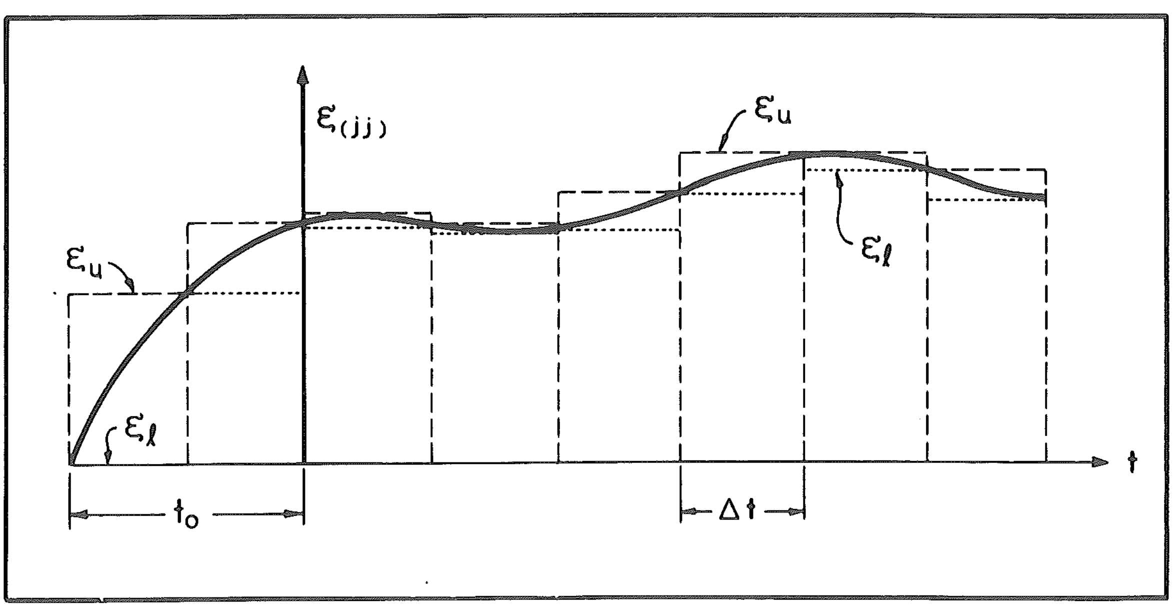


Figure 3. Multi step-function approximations of strain loading.

$$\sigma_{(ij)}(t) = \left[\phi(o)(\ ) + \int_0^t (\ ) \frac{\partial\phi(t-\tau)}{\partial\tau} d\tau \right] \varepsilon_{(ij)}(t) \tag{11}$$

Comparing Eqs. (5) and (11) shows that $R(t)$ may be defined by Eq. (12),

$$R(t) = \phi(o)(\ ) + \int_0^t (\ ) \frac{\partial\phi(t-\tau)}{\partial\tau} d\tau \tag{12}$$

if the deformation is quasi-static. Substituting Eq. (12) in Eq. (4) yields the form of constitutive equations that are used to describe the viscoelastic plate material in this study

$$\phi(0)\,\varepsilon_{ij}(t) + \int_0^t \varepsilon_{ij}(\tau) \frac{\partial\phi(t-\tau)}{\partial\tau} d\tau = \frac{3}{2}\sigma_{ij} - \frac{\sigma_{mm}}{2}\,\delta_{ij} \tag{13}$$

VI

VISCOELASTIC THIN PLATE THEORY

6.1 The Viscoelastic Plate Deflection Equation

In order to define the deformation response of the plate as a whole to the externally applied load and the foundation reaction, it is necessary to insure point-to-point equilibrium throughout the plate continuum. The equilibrium equations for the viscoelastic continuum are given by Eq. (1):

$$\sigma_{ij,i} = 0 \tag{1}$$

where σ_{ij} is the stress tensor.

The constitutive equations (13) from Chapter V, equilibrium equations (1), and small strain definitions can be used to derive a viscoelastic plate bending equation which is analogous to the differential equation that describes the bending deformation of thin, perfectly elastic plates. In the derivation that follows it is assumed that:

1) the plate surfaces are subjected to normal stress tractions only

2) there are no forces applied in the plane of the plate

3) the body forces are equal to zero

4) the vertical plate deflection w is small in comparison to the plate thickness

5) the plate thickness is small in comparison to its in-plane dimensions.

Restrictions 1) through 5) above, lead to the following assumptions concerning the form of the plate displacement field $\vec{u}$.

1) Lines normal to the mid-surface of the plate are assumed to remain normal to the mid-surface during the deformation (shearing deformation is neglected).

2) The mid-surface of the plate is assumed to remain unstrained during the deformation.

3) It is assumed that the normal stresses perpendicular to the plate surface do not significantly affect the strains in the plane of the plate.

These assumptions yield displacement component functions of the form (2):

$$u = -z\frac{\partial w}{\partial x} \;,\; v = -z\frac{\partial w}{\partial y} \tag{2}$$

where u and v are the displacements occurring in the plane of the plate and w is the vertical displacement component. The strain components can be computed using Eq. (2), as shown in Eq. (3):

$$\varepsilon_{xx} = -z\frac{\partial^2 w}{\partial x^2} \;,\; \varepsilon_{yy} = -z\frac{\partial^2 w}{\partial y^2} \;,\; \varepsilon_{xy} = -z\frac{\partial^2 w}{\partial x \partial y} \tag{3}$$

Next, holding time t fixed, but arbitrary, and integrating the equilibrium equations (1) over the thickness of the plate yields three resultant equilibrium conditions which must be satisfied. Two of the resultant equations are automatically satisfied, the third is given by Eq. (4):

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q^+ - q^- = 0 \tag{4}$$

where Q_x and Q_y are the resultants of the σ_{xx} and σ_{yy} stresses, and q^+ and q^- are the stress tractions applied to the upper and lower faces of the plate. Two additional resultant equations can be obtained by computing the moments of the first two equilibrium equations about the middle plane of the plate; then computing the resultants of these equations over the depth of the plate. The resultant moment expressions are given in Eq. (5):

$$\frac{\partial M_x}{\partial x} - \frac{\partial M_{xy}}{\partial y} - Q_x = 0$$

$$- \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y = 0 \tag{5}$$

where
$$M_{xy} \equiv \int_{-h/2}^{h/2} \sigma_{yx} \, z \, dz \ , \ M_x \equiv \int_{-h/2}^{h/2} \sigma_{xx} \, z \, dz \ , \ M_y \equiv \int_{-h/2}^{h/2} \sigma_{yy} \, z \, dz$$

As in the case of thin elastic plate theory, displacement solutions of the form Eq. (2) are sought such that equilibrium is satisfied in the resultant sense as defined by Eqs. (4), (5), at all times t. Eliminating Q_x and Q_y from Eqs. (4) and (5) yields Eq. (6):

$$\frac{\partial^2 M_x}{\partial x^2} - \frac{2\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} = q^- - q^+ \tag{6}$$

one single differential equation which must be satisfied to insure equilibrium in the resultant sense.

Equation (6) can be expressed in terms of the vertical displacement component w by substituting the strain equations (3) into the constitutive equations (13) of Chapter V and computing M_x , M_{xy} , and M_y from the resulting expressions. In doing this, assumption 3) is used to simplify the constitutive relationships. Also, t is held fixed, but arbitrary, for these operations. Substituting the resultant expressions into Eq. (6) yields the thin plate equation of viscoelasticity in Cartesian co-ordinates.

$$\frac{R(t)h^3}{12} \left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right) = q^+ - q^- \tag{7}$$

Using symbolic notation the plate equation takes the form Eq. (8):

$$D(t) \nabla^2 \nabla^2 w = q^+ - q^-, \ D(t) \equiv \frac{R(t)}{q} h^3 \equiv D R(t) \tag{8}$$

in any admissable co-ordinate system.

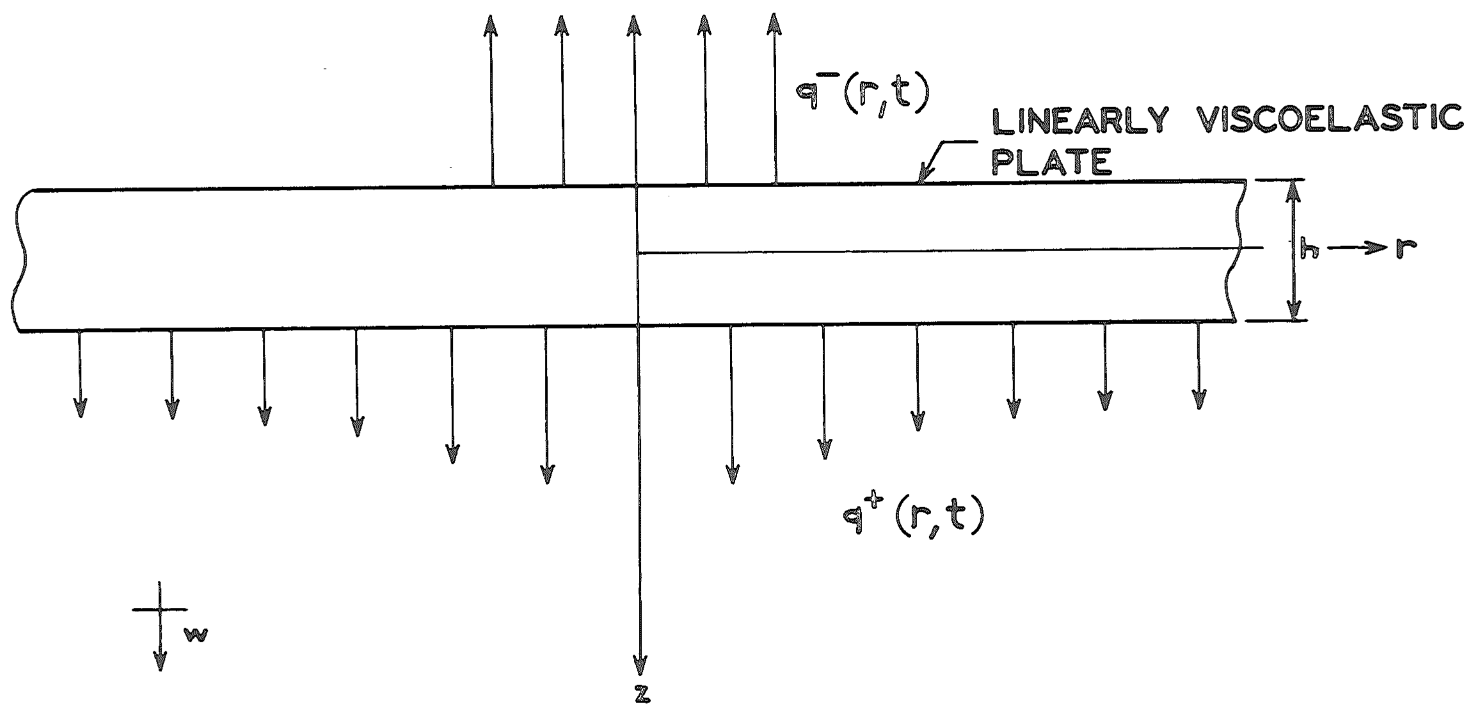


Figure 4. The viscoelastic plate.

6.2 Solution of Plate Equation

In this section, an iterated Laplace-Hankel transformation is used to solve for the transform image of the plate deflection in terms of the transformed foundation reaction. This provides a boundary condition for the poro-elastic foundation boundary value problem. With this expression, sufficient boundary conditions are known to define the foundation boundary value problem.

Considering the mathematical model that is being analyzed, Eq. (8) is restricted to a cylindrical co-ordinate system, and w is assumed to be a function of radius r and time t only. Figure 4 illustrates the geometry of the plate, the reference co-ordinate system and the surface traction sign convention. The z axis in Figure 4 extends downward into the half-space foundation.

The plate equation (8) takes the form Eq. (9):

$$D(t)\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)w(r,t) = q^+(r,t)-q^-(r,t) \tag{9}$$

where,

$$q^-(r,t) = \begin{cases} H(t)\,q\ , & r<b \\ 0\ , & r>b \end{cases} \tag{10}$$

and

$$H(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t<0 \end{cases} \tag{11}$$

in the cylindrical co-ordinate system for the axially symmetric loading shown in Figure 4. The bending stiffness $D(t)$ which is defined in Eq. (8) contains the time operator $R(t)$ from Eq. (12) of Chapter V.

There are two unknowns in Eq. (9); the plate deflection w and the reactive pressure q^+ on the lower face of the plate. An additional equation relating these two unknowns will be obtained subsequently by considering the deformation response of the half-space of poro-elastic material which supports the loaded plate.

It is useful to modify Eq. (9) so that the partial differentials in r can be eliminated [9, 10]. This can be done by utilizing the zero order Hankel integral transformation that is defined by Eq. (12):

$$f_0(\eta) = \int_0^\infty r\,f(r)\,J_0(\eta r)\,dr \tag{12}$$

Applying this transformation to both sides of Eq. (9) and rearranging terms yields Eq. (13):

$$D\left[\Phi(0)\,w_0(\eta,t) + \int_0^t w_0(\eta,t)\,\frac{\partial\Phi(t-\tau)}{\partial\tau}\,d\tau\right] = \frac{q}{\eta^5}\,b\,J_1(b\eta)\,H(t) + q_0^+(\eta,t)/\eta^4 \tag{13}$$

for the transformed displacement w_0 .

Further simplification can be achieved by applying the Laplace integral transformation [9, 10], defined by Eq. (14):

$$\bar{f}(s) = \int_0^\infty e^{-st}\,f(t)\,dt \tag{14}$$

to Eq. (13). The final result is given in Eq. (15):

$$s\,D\,\bar{\Phi}(s)\,\bar{w}_0(\eta,s) = \frac{q\,b\,J_1(b\eta)}{\eta^5 s} + \frac{\bar{q}_0^+(\eta,s)}{\eta^4} \tag{15}$$

which is an algebraic equation in η and s relating $\bar{w}_0$ and $\bar{q}_0^+$. Solving Eq. (15) for $\bar{q}_0^+$ yields Eq. (16):

$$\bar{q}_0^+ = \eta^4 s\,D\bar{\Phi}(s)\,\bar{w}_0(\eta,s) - q\,\frac{b\,J_1(b\eta)}{\eta\,s} \tag{16}$$

which may be regarded as a boundary condition on the foundation at the plate - foundation interface. It is noted that although the plate equation has been reduced to an algebraic equation that relates $\bar{q}_0^+$ and $\bar{w}_0$, neither of the unknowns $\bar{q}_0^+$ and $\bar{w}_0$ can be determined explicitly from this single equation. It is, therefore, necessary to investigate the foundation-plate interaction as well, in order to determine $\bar{q}_0^+$ and $\bar{w}_0$.

VII

DEFINING THE PLATE ON HALF-SPACE PROBLEM

There are two unknowns in Eq. (16) of 6.2, the transformed plate deflection $\bar{w}_0$ and the transformed reactive pressure $\bar{q}_0^+$. An additional equation relating these two unknowns can be obtained by considering the response of the foundation that lies below the plate. This can be done by formulating an initial-boundary value problem in the foundation. In doing this it is convenient to use a cylindrical co-ordinate system because the loading and deformation are axially symmetric.

7.1 Boundary Conditions

The constitutive equations for the solid partial stress tensor given in Eq. (36) of 3.4 can be expressed in cylindrical co-ordinates in terms of physical components as shown in Eq. (1):

$$\begin{aligned}
\sigma_{rr} &= -a_1 p + 2a_2 \frac{\partial u_r}{\partial r} + a_3\left(\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z}\right) \\
\sigma_{\theta\theta} &= -a_1 p + 2a_2 \frac{u_r}{r} + a_3\left(\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z}\right) \\
\sigma_{rz} &= a_2\left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right) \\
\sigma_{zz} &= -a_1 p + 2a_2 \frac{\partial u_z}{\partial z} + a_3\left(\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z}\right)
\end{aligned} \tag{1}$$

In order to determine the boundary conditions on $\mathcal{E}$ and $\mathcal{S}$ it is necessary to express the stresses in terms of $\mathcal{E}$ and $\mathcal{S}$. Substituting Eqs. (24), (28), and (29) of 4.2 in Eq. (1) yields expressions for the stress components in terms of functions E and S.

$$\sigma_{rr} = \frac{(a_3 - 2a_1a_2)}{(1+a_1)} \nabla^2 E + 2a_2 \left(\frac{\partial^2 E}{\partial r^2} + z \frac{\partial^2 S}{\partial r^2} - \frac{a_1}{(1+a_1)} \frac{\partial S}{\partial z} \right)$$

$$\sigma_{zz} = \frac{(a_3 - 2a_1a_2)}{(1+a_1)} \nabla^2 E + 2a_2 \left(\frac{\partial^2 E}{\partial z^2} + z \frac{\partial^2 S}{\partial z^2} - \frac{a_1}{(1+a_1)} \frac{\partial S}{\partial z} \right)$$

$$\sigma_{\theta\theta} = \frac{(a_3 - 2a_1a_2)}{(1+a_1)} \nabla^2 E + 2a_2 \left(\frac{1}{r} \frac{\partial E}{\partial r} + \frac{z}{r} \frac{\partial S}{\partial r} - \frac{a_1}{(1+a_1)} \frac{\partial S}{\partial z} \right) \tag{2}$$

$$\sigma_{rz} = 2a_2 \left(\frac{\partial^2 E}{\partial r \partial z} + z \frac{\partial^2 S}{\partial r \partial z} \right) \qquad \sigma_{r\theta} = \sigma_{z\theta} = 0$$

The boundary conditions can be determined by considering the state of stress at the boundaries. At $z = 0$ the total stress in the z direction on the z face of an element of the poro-elastic material must equal q^+, the plate reaction. This means that the traction stress q^+ is exerted on both the fluid and solid components of the solid-fluid mixture. Using the definitions of 4.2, this boundary condition can be expressed in terms of E and S as shown in Eq. (3).

$$q^+(r,t) = 2a_2 \left(\frac{\partial^2 E}{\partial z^2} - \frac{\partial S}{\partial z} - \nabla^2 E \right) \Bigg|_{z=0} \qquad t > 0 \tag{3}$$

It is assumed that the traction q^+ at the surface of the half-space vanishes as r becomes large. This is physically apparent since q^+ is the reaction that results under the plate when a circular uniform load of finite radius is applied to the top surface of the plate.

The surface at $z = 0$ is assumed to be impermeable to the fluid at any time t. Therefore, the relative velocity vector of the fluid to the solid material must be zero in the z direction at the surface. Equation (10) of 4.1 states that the partial derivative of p with respect to z is proportional to the relative velocity in the z direction and, therefore, is zero at $z = 0$. This condition is expressed by Eq. (4).

$$\frac{2a_2}{1+a_1}\left(\frac{\partial^2 S}{\partial z^2} + \frac{(a_3+2a_2)}{2a_2}\nabla^2 \frac{\partial E}{\partial z}\right)\Big|_{z=0} = 0 \qquad \begin{matrix} t>0 \\ r\geq 0 \end{matrix} \tag{4}$$

The shear traction σ_{zr} at $z=0$ is equal to zero, since it is assumed that the plate is free to deform horizontally relative to the half-space surface; this condition is expressed by Eq. (5):

$$2a_2\left(\frac{\partial^2 E}{\partial r \partial z}(r,o,t)\right) = 0 \qquad t>0,\ r\geq 0 \tag{5}$$

With regard to displacements, it is assumed that the vertical displacements in the foundation at $z=0$ are equal to the plate deflections there, as shown in Eq. (6).

$$u(r,o,t) = w(r,t) \tag{6}$$

It is also assumed that the stresses, and displacements, and rigid rotation in the half-space vanish as $\sqrt{r^2+z^2}$ approaches infinity. These conditons are expressed in the following equations:

$$\begin{matrix} \lim\limits_{\sqrt{r^2+z^2}\to\infty} \sigma_{\theta\theta} = 0 & \lim \sigma_{rz} = 0 & \lim \sigma_{rr} = 0 \\ \lim p = 0 & \lim \sigma_{zz} = 0 & \\ \lim u_r = 0 & \lim u_z = 0 & \\ \lim \frac{\partial S}{\partial r} = 0 & & \end{matrix} \tag{7}$$

By considering Eq. (2) of this section and Eqs. (24), (28), and (29) of 4.2, it can be seen that the above physical conditions are satisfied if the following restrictions are placed on E and S :

$$\begin{matrix} \lim\limits_{\sqrt{r^2+z^2}\to\infty} \nabla^2 E = 0 & \lim \frac{\partial S}{\partial z} = 0 & \lim\left(\frac{\partial^2 E}{\partial z^2} + z\frac{\partial^2 S}{\partial z^2}\right) = 0 \\ \lim\left(\frac{\partial E}{\partial r} + z\frac{\partial S}{\partial r}\right) = 0 & \lim\left(\frac{\partial^2 E}{\partial r^2} + z\frac{\partial^2 S}{\partial r^2}\right) = 0 & \\ \lim\left(\frac{\partial E}{\partial z} + \frac{\partial S}{\partial z} - S\right) = 0 & \lim\left(\frac{\partial^2 E}{\partial r\partial z} + z\frac{\partial^2 S}{\partial r \partial z}\right) = 0 & \\ \lim \frac{\partial S}{\partial r} = 0 & & \end{matrix} \tag{8}$$

When the step function load is applied at time $t = 0$, the plate will respond with an instantaneous elastic deformation, and the skeletal structure of the porous media foundation will undergo an instantaneous volume change because the skeletal material is assumed to be compressible. This means that at time zero e_{mm} takes on an initial value which places a restriction on the function $E(r,z,t)$ as shown in Eq. (9).

$$e_{mm}(r,z,o) = \nabla^2 E(r,z,o) = f(r,z) \qquad (9)$$

Equation (9) is the initial condition for partial differential (25) of 4.2. The function $f(r,z)$ can be determined by considering an elastic plate, with a Youngs modulus equal to $R(o)$ supported on an elastic half-space that has Lamé constants a_2 and $(a_4 - {}^{(2)}\bar{\rho}\, a_8)$. The initial condition is developed explicitly in 7.2 and is stated here for reference:

$$\nabla^2 E(r,z,o) = \frac{1}{(a_4 - {}^{(2)}\bar{\rho} a_8 + a_2)} \int_o^\infty C q e^{-\eta z} J_1(b\eta) J_0(\eta r)\, d\eta \qquad (10)$$

where:

$$C = -\frac{2a_2(a_4 - {}^{(2)}\bar{\rho} a_8 + a_2) b}{D R(o)(a_4 - {}^{(2)}\bar{\rho} a_8 + 2a_2)\eta^3 + 2a_2(a_4 - {}^{(2)}\bar{\rho} a_8 + a_2)} \qquad (11)$$

7.2 Initial Condition on $\nabla^2 E$

In order to solve for $\nabla^2 E = e_{mm}$ in partial differential equation (25) of 4.2, it is necessary to know the initial value of e_{mm}. At $t = 0$ the constitutive equations for the partial solid and fluid stresses reduce to Eq. (37) of 3.4 which may be added together to give a relationship for the total stresses as shown in Eq. (12).

$$\sigma_{ij} + \pi_{ij} = 2a_2 e_{ij} + (a_4 - {}^{(2)}\bar{\rho} a_8) e_{mm} \delta_{ij} \qquad (12)$$

In addition to satisfying the constitutive equations (12), $\sigma_{ij} + \pi_{ij}$ must also satisfy the equilibrium equations (13) of 3.2 at time zero.

$$(\sigma_{ij} + \pi_{ij})_{,i} = 0 \qquad (13)$$

Therefore, determining the initial condition on $\nabla^2 E = e_{mm}$ reduces to finding the solution to the analogous 'elasticity' problem defined by the total stress tensor constitutive equations (12), the equilibrium equations (13), the Beltrami-Michell compatibility conditions [11], and appropriate boundary conditions.

The boundary conditions on $\sigma_{ij} + \pi_{ij}$ can be determined by considering the magnitude of the stresses at time zero, at the surface of the half-space. The total normal stress applied to the top surface of poro-elastic half-space at time zero is $q^+(r,o)$ as expressed in Eq. (14).

$$\sigma_{zz}(r,o,o) + \pi_{zz}(r,o,o) = q(r,o)^+ \tag{14}$$

According to the assumed frictionless boundary condition at the plate - half-space interface, the shear stresses vanish there.

$$\sigma_{rz}(r,o,o) + \pi_{rz}(r,o,o) = 0 \tag{15}$$

In order to obtain an expression for $q^+(r,o)$, the Hankel transformed plate equation (13) of 6.2 can be considered in the following modified form, which results when the time t is set equal to zero.

$$DR(o)w_0(\eta,o) = \frac{q}{\eta^5} bJ_1(b\eta) + \frac{q_0^+(\eta,o)}{\eta^4} \tag{16}$$

The unknown transformed plate deflection w_0 in Eq. (16) is assumed to be equal to the transformed vertical deflection of the half-space at $z=0$. This condition is stated in Eq. (17).

$$w_0(\eta,o) = u_{z_0}(\eta,o,o) \tag{17}$$

The deflection $u_z(r,o)$ resulting from the axially symmetric normal surface traction at $z=0$ on the half-space can be computed using the solution of Terrazawa [12] as shown in Eq. (18).

$$u_z(r,z,o)\Big|_{z=0} = -\frac{(a_4 - {}^{(2)}\bar{\rho}a_8 + a_2)}{2a_2(a_4 - {}^{(2)}\bar{\rho}a_8 + a_2)} \int_0^\infty \frac{Z(\eta)J_0(\eta r)\,d\eta}{\eta} \tag{18}$$

where

$$\frac{Z(\eta)}{\eta} = q_0^+(\eta,o)$$

Taking the zero order Hankel transform of Eq. (18) and substituting for the transformed plate deflection w_0 in Eq. (16), results in Eq. (19) for q_0^+ at time zero.

$$q_0^+(\eta,o) = -\frac{DR(o)\eta^2(a_4 - {}^{(2)}\bar{\rho}\, a_8 + 2a_2)\, Z(\eta)}{2a_2(a_4 - {}^{(2)}\bar{\rho}\, a_8 + a_2)} - \frac{qbJ_1(b\eta)}{\eta} \tag{19}$$

An explicit expression of $Z(\eta)$ is desired since $\sigma_{mm} + \pi_{mm}$ can be computed [12] if $Z(\eta)$ is known. Substituting the second equality (18) in (19) solving for $Z(\eta)$ provides the desired result:

$$Z(\eta) = CqJ_1(b\eta) \tag{20}$$

where

$$C = \frac{-2a_2(a_4 - {}^{(2)}\bar{\rho} a_8 + a_2)b}{DR(o)\eta^3(a_4 - {}^{(2)}\bar{\rho} a_8 + 2a_2) + 2a_2(a_4 - {}^{(2)}\bar{\rho} a_8 + a_2)}$$

Finally, using the Terrazawa solution again, with Eq. (20) substituted for $Z(\eta)$, the solution of $\sigma_{mm} + \pi_{mm}$ is obtained.

$$(\sigma_{mm} + \pi_{mm})\Big|_{t=0} = \frac{(3a_4 - 3{}^{(2)}\bar{\rho} a_8 + 2a_2)}{(a_4 - {}^{(2)}\bar{\rho} a_8 + a_2)} \int_0^\infty CqJ_1(b\eta)e^{-\eta z} J_0(\eta r)\, d\eta \tag{21}$$

By examining the constitutive equations (12) it is apparent that $e_{mm}(r,z,o)$ is related to $\sigma_{mm} + \pi_{mm}$ by Eq. (22).

$$(\sigma_{mm} + \pi_{mm})\Big|_{t=0} = (2a_2 + 3a_4 - 3{}^{(2)}\bar{\rho} a_8)\, e_{mm}(r,z,o) \tag{22}$$

Equating Eq. (21) and Eq. (22) and solving for $e_{mm}(r,z,o)$ yields the desired initial condition as given by Eq. (23):

$$\nabla^2 E(r,z,o) = e_{mm}(r,z,o) = \int_0^\infty \frac{Cqe^{-\eta z} J_1(b\eta) J_0(\eta r)\, d\eta}{(a_4 - {}^{(2)}\bar{\rho} a_8 + a_2)} \tag{23}$$

where the left term comes from the derivation of E (Eq. (26) of 4.2). If this equation is operated on by the zero order Hankel transform, condition (24) results.

$$e_{mm_0}(\eta,z,o) = \frac{1}{(a_4 - a_8^{(2)}\bar{\rho} + a_2)} \frac{Cqe^{-\eta z} J_1(b\eta)}{\eta} \tag{24}$$

The transformed quantity $e_{mm_0}(\eta,z,o)$ provides an initial condition in mathematical terms that accounts for the instantaneous volume change that occurs in the foundation upon application of external loading. This condition is necessary to the solution of the partial differential equation that defines E .

7.3 The Initial-Boundary Value Problem

The significance of the results obtained in Chapter VII can be summarized by outlining the initial-boundary value problem. First, two partial differential equations must be solved.

$$c\nabla^4 E(r,z,t) = \nabla^2 \frac{\partial E}{\partial t}(r,z,t) \qquad (25)$$

$$\nabla^2 S(r,z,t) = 0$$

where

$$z > 0, \; r \geq 0, \; t \geq 0$$

These governing equations are from 4.2. The solution of the first equation in (25) will be required to satisfy the following initial condition:

$$\nabla^2 E(r,z,0) = f(r,z) \qquad (26)$$

The solutions E and S must also satisfy boundary conditions (27) through (29) at $z = 0$.

$$\frac{\partial^2 E}{\partial r \partial z}(r,0,t) = 0 \qquad r \geq 0, \; t > 0 \qquad (27)$$

$$2a_2\left(\frac{\partial^2 E}{\partial z^2} - \frac{\partial S}{\partial z} - \nabla^2 E\right) = q^+(r,t) \qquad r \geq 0, \; z = 0, \; t > 0 \qquad (28)$$

$$\frac{\partial^2 S}{\partial z^2} + \frac{(a_3 + 2a_2)}{2a_2}\nabla^2 \frac{\partial E}{\partial z}) = 0 \qquad r \geq 0, \; z = 0, \; t > 0 \qquad (29)$$

The solutions will be required to satisfy boundary conditions (30) also.

$$u(r,0,t) = w(r,t) \qquad (30)$$

As $\sqrt{r^2 + z^2}$ approaches infinity (31) must be satisfied.

$$\lim \nabla^2 E = 0 \quad \lim \frac{\partial S}{\partial z} = 0 \quad \lim\left(\frac{\partial^2 E}{\partial z^2} + z\frac{\partial^2 S}{\partial z^2}\right) = 0 \quad \lim\left(\frac{\partial E}{\partial r} + z\frac{\partial S}{\partial r}\right) = 0$$

$$\sqrt{r^2 + z^2} \to \infty \qquad (31)$$

$$\lim\left(\frac{\partial^2 E}{\partial r^2} + z\frac{\partial^2 S}{\partial r^2}\right) = 0 \quad \lim\left(\frac{\partial E}{\partial z} + \frac{\partial S}{\partial z} - S\right) = 0 \quad \lim\left(\frac{\partial^2 E}{\partial r \partial z} + z\frac{\partial^2 S}{\partial r \partial z}\right) = 0$$

$$\lim \frac{\partial S}{\partial r} = 0$$

A method will be developed for solving the system of Eqs. (25) through (31) in Chapter VIII. Generally, the method will consist of applying integral transformations to the problem in order to reduce partial equations (12) to ordinary differential equations having one independent variable z. Transformed initial and boundary conditions will then be imposed and transformed solutions for the stresses and displacements will be established.

VIII

SOLUTION OF INITIAL-BOUNDARY VALUE PROBLEM

The problem has now been defined and reduced to terms that allow solution. The initial-boundary value problem that defines the foundation deformation is summarized in 7.3. In 8.1, the governing partial differential equations will be solved using iterated Laplace-Hankel transformations. In 8.2, boundary conditions will be transformed and imposed on the solutions for $\bar{E}_0$ and $\bar{S}_0$ obtained in 8.1 and transformed solutions for the plate deflection, foundation reaction, and several other variables will be obtained. Finally, in 8.3 the transformed solutions of the remaining unknowns are obtained.

8.1 Solution of Differential Equations Using Transform Methods

The first differential equation (25) of 7.3 takes the form (1):

$$c\nabla^2 e_{mm} = \frac{\partial e_{mm}}{\partial t} \tag{1}$$

when $\nabla^2 E$ is replaced by e_{mm}. Taking the Laplace transformation of Eq. (1) eliminates the partial derivative with respect to t and reduces (1) to a partial differential equation in r and z.

$$c\nabla^2 \bar{e}_{mm}(r,z,s) = s\bar{e}_{mm}(r,z,s) - e_{mm}(r,z,o) \tag{2}$$

Equation (3) is the Hankel transform of (2):

$$\left(\frac{d^2}{dz^2} - \eta^2\right)\bar{e}_{mm_0}(\eta,z,s) = \frac{s}{c}\bar{e}_{mm_0}(\eta,z,s) - \frac{e_{mm_0}(\eta,z,o)}{c} \tag{3}$$

which contains derivatives with respect to z only. Substituting the initial condition expression previously obtained for $e_{mm_0}(\eta,z,o)$ in 7.2 results in Eq. (4).

$$\left(\frac{d^2}{dz^2}-(\eta^2+\frac{s}{c})\right)\bar{e}_{mm_0}=ke^{-\eta z} \tag{4}$$

where $$k=-\frac{1}{(a_4-{}^{(2)}\bar{\rho}a_8+a_2)}\,Cq\frac{1}{\eta c}J_1(b\eta)$$

The general solution of the ordinary differential equation (4) is given by Eq.(5):

$$\bar{e}_{mm_0}(\eta,z,s)=B_1e^{-\eta z}+B_2e^{-\mu z}+B_3e^{+\mu z} \tag{5}$$

where

$$B_1=-k\frac{c}{s}\ ,\quad B_2=d_1+\frac{k}{2\mu(\mu+\eta)}\ ,\quad B_3=d'_1+\frac{k}{2\mu(\mu+\eta)}\ ,$$

d_1 and d_1' are abritrary functions of η and s

and $\mu=\sqrt{\eta^2+s/c}$.

From the definition of the displacement function $E(r,z,t)$ it has been shown that $\bar{e}_{mm_0}(\eta,z,s)$ may be expressed by Eq. (6):

$$\left(\frac{d^2}{dz^2}-\eta^2\right)\bar{E}_0(\eta,z,s)=\bar{e}_{mm_0}(\eta,z,s) \tag{6}$$

Substituting Eq. (6) in Eq. (5) yields the differential equation that defines $\bar{E}_0(\eta,z,s)$.

$$\left(\frac{d^2}{dz^2}-\eta^2\right)\bar{E}_0(\eta,z,s)=B_1e^{-\eta z}+B_2e^{-\mu z}+B_3e^{+\mu z} \tag{7}$$

Equation (7) is an ordinary, non-homogeneous differential equation and its general solution is given below:

$$\bar{E}_0(\eta,z,s)=(C_1+d_2)e^{-\eta z}+(C_2+d_2')e^{\eta z}+C_3ze^{-\eta z}+C_4e^{\mu z}+C_5e^{-\mu z} \tag{8}$$

where

$$C_1=\left(-\frac{B_1}{4\eta^2}+\frac{B_2}{2\eta(\mu-\eta)}-\frac{B_3}{2\eta(\mu+\eta)}\right)$$

$$C_2=\left(-\frac{B_1}{4\eta^2}-\frac{B_2}{2\eta(\mu-\eta)}+\frac{B_3}{2\eta(\mu-\eta)}\right)$$

$$C_3=\frac{kc}{2\eta s}\ ,\quad C_4=\frac{B_3c}{s}\ ,\quad C_5=\frac{B_2c}{s}$$

and d_2, d_2' are arbitrary functions of η and s .

The second Eq. (25) of 7.3 is the governing differential equation for the displacement generating function S. Applying the Laplace and then zero order Hankel transformations to that equation gives Eq. (9):

$$\left(\frac{d^2}{dz^2} - \eta^2\right) \bar{S}_0 (\eta, z, s) = 0 \tag{9}$$

which is a homogeneous ordinary differential equation in z. The general solution to this equation is given by Eq. (10):

$$\bar{S}_0 (\eta, z, s) = d_3 e^{-\eta z} + d_3' e^{\eta z} \tag{10}$$

where d_3 and d_3' are arbitrary functions of η and s.

This completes the solution of the differential equations that define $\bar{\bar{E}}_0$ and $\bar{\bar{S}}_0$. Equation (8) is the solution for $\bar{\bar{E}}_0$, and Eq. (10) is the solution for $\bar{\bar{S}}_0$. There are unknown coefficients multiplying each term in Eqs. (8) and (10). These coefficients are constant with respect to z, but must be regarded as functions of η and s. In 8.2 the transformed boundary conditions will be imposed on the solutions $\bar{\bar{E}}_0$ and $\bar{\bar{S}}_0$, and this will result in the unknown coefficient functions becoming fully defined.

8.2 Determining Laplace-Hankel Transforms

If the first three boundary conditions at $z = \infty$ given in Eq. (31) in 7.3 are now invoked; the general solutions $\bar{\bar{E}}_0$ and $\bar{\bar{S}}_0$ can be simplified. The first of the conditions implies that $\bar{\bar{e}}_{mm_0}$ must approach zero as z approaches infinity. Therefore, the coefficient B_3 in Eq. (5) must vanish. Then, according to the definition of C_4 given in Eq. (8), C_4 must also vanish.

$$B_3 = C_4 = 0 \tag{11}$$

The second boundary condition at $z = \infty$ implies that $\frac{\partial \bar{\bar{S}}_0}{\partial z}$ must vanish as z approaches infinity. This condition makes it necessary that d_3' be equal to zero as shown in Eq. (12):

$$\lim_{z\to\infty} \frac{\partial \bar{S}_0}{\partial z}(\eta,z,s) = \eta d'_3\, e^{\eta z} = 0 \tag{12}$$

Therefore, Eq. (10) reduces to Eq. (13):

$$\bar{S}_0(\eta,z,s) = d_3 e^{-\eta z} \tag{13}$$

The constraint on the transformed quantities that results from the third condition is given in Eq. (14).

$$\lim_{z\to\infty} \left(\frac{\partial^2 \bar{E}_0}{\partial z^2} + z\frac{\partial^2 \bar{S}_0}{\partial z^2}\right) = 0 \tag{14}$$

Substituting from Eqs. (8) and (13) and invoking Eq. (11) results in Eq. (15):

$$\lim_{z\to\infty} \left(\eta^2 (C_2 + d'_2)\, e^{\eta z} + \eta^2 C_3 z e^{-\eta z} + \eta^2 d_3 e^{-\eta z}\right) = 0 \tag{15}$$

For Eq. (15) to be true, the constants C_2 and d'_2 must satisfy condition (16):

$$(C_2 + d'_2) = 0 \tag{16}$$

Now consider the boundary conditions at $z=0$.[7] If Eqs. (27) and (29) of 7.3 are operated on by the repeated Laplace-Hankel transformations, the two conditions given in Eq. (17) result.

$$-\eta \frac{d\bar{E}_0}{dz}(\eta, o, s) = 0$$

$$\frac{d^2\bar{S}_0}{dz^2}(\eta,o,s) + \frac{(a_3 + 2a_2)}{2a_2}\left(\frac{d^2}{dz^2} - \eta^2\right)\frac{d\bar{E}_0}{dz}(\eta,o,s) = 0 \tag{17}$$

Substituting for $\bar{E}_0$ in the first Eq. (17) results in the following expression:

$$\eta\,(C_1 + d_2) = C_3 - \mu C_5 \tag{18}$$

An expression can be obtained from the second Eq. (17) by substituting for both $\bar{E}_0$ and $\bar{S}_0$.

$$d_3 = -\frac{(a_3 + 2a_2)}{2a_2\eta^2}\left(2\eta^2 C_3 + \mu(\eta^2 - \mu^2) C_5\right) \tag{19}$$

Taking the Laplace-Hankel transformation of boundary condition (28) in 7.3 yields Eq. (20):

[7] The remaining conditions (31) of 7.3 are satisfied.

$$-2a_2\left(\frac{d\bar{S}_0}{dz}(\eta,o,s)-\eta^2\,\bar{E}_0(\eta,o,s)\right)=\bar{q}_0^+ \qquad (20)$$

Substituting for $\bar{E}_0$ and $\bar{S}_0$ gives Eq. (21).

$$-2a_2\left(-\eta d_3-\eta^2((C_1+d_2)+C_5)\right)=\bar{q}_0^+ \qquad (21)$$

The Laplace-Hankel transformed plate equation (15) of Chapter VI is repeated here as Eq. (22).

$$\bar{w}_0(\eta,s)\,Ds\bar{\phi}(s)=\frac{q}{\eta^5 s}\,bJ_1(b\eta)+\frac{\bar{q}_0^+(\eta,s)}{\eta^4} \qquad (22)$$

The plate deflection is assumed to equal the half-space deflection at $z=0$ for all $t\geq 0$. Therefore, $\bar{w}_0$ can be related to $\bar{E}_0$ and $\bar{S}_0$ as follows, using Eq. (29) of 4.2:

$$\bar{w}_0(\eta,s)=\bar{u}_{z_0}(\eta,o,s)=\frac{d\bar{E}_0}{dz}(\eta,o,s)-\bar{S}_0(\eta,o,s) \qquad (23)$$

It then follows by substitution for $\bar{E}_0$ and $\bar{S}_0$ from Eqs. (8) and (13) that Eq. (24) is valid.

$$\bar{w}_0(\eta,s)=-\eta(C_1+d_2)+C_3-\mu C_5-d_3 \qquad (24)$$

Substituting Eq. (24) into the plate equations (22) and making use of Eq. (18) yields Eq. (25).

$$\bar{q}_0^+=-\eta^4 Ds\bar{\phi}(s)\,d_3-\frac{qbJ_1(b\eta)}{\eta s} \qquad (25)$$

Solving Eqs. (18), (19), and (21) simultaneously yields an expression for d_3.

$$d_3=\frac{C_6-\bar{q}_0^+}{C_7} \qquad (26)$$

where

$$C_6=\frac{ka_2(2\eta+\mu)}{\mu(\mu+\eta)^2}$$

and

$$C_7=\frac{(2a_2)^2\eta^3-2a_2\eta^2\mu(a_3+2a_2)-2a_2\,\eta\mu^2(a_3+2a_2)}{\mu(\mu+\eta)(a_3+2a_2)}$$

Substitution of this result in Eq. (25) yields the following transformed solution for the reactive pressure $\bar{q}_0^+$ under the plate.

$$\bar{q}_0^+=\frac{-\eta^5 s^2\;D\bar{\phi}(s)C_6-C_7\,qbJ_1(b\eta)}{\eta s(C_7-\eta^4 Ds\bar{\phi}(s))} \qquad (27)$$

Substitution of expression (18) for $\eta(C_1+d_2)$ in Eq. (24) simplifies the transformed plate deflection expression to the following.

$$\bar{w}_0(\eta, s) = -d_3 \tag{28}$$

Substituting the expression obtained for d_3 into Eq. (28) yields the transformed solution for the plate deflection.

$$\bar{w}_0(\eta, s) = \frac{\bar{q}_0^+ - C_6}{C_7} \tag{29}$$

The solution $\bar{S}_0$ follows directly from Eq. (13):

$$\bar{S}_0(\eta, z, s) = \frac{(C_6 - \bar{q}_0^+)}{C_7} e^{-\eta z} \tag{30}$$

Now referring to Eq. (8) for $\bar{E}_0$; the constant $C_1 + d_2$ is related to the known constants C_3 and C_5 through Eq. (18). Also, $(C_2 + d_2)$ and C_4 are equal to zero by Eqs. (16) and (11), respectively. Therefore, in order to define $\bar{E}_0$ completely it is only necessary to determine C_5 . Equation (21) can now be used to solve for C_5 since $\bar{q}_0^+$ is known by Eq. (27). Substituting for $(C_1 + d_2)$ from Eq. (18) into (21) and solving for C_5 results in the following expression:

$$C_5 = \frac{\bar{q}_0^+ - 2a_2\eta\,(d_3 + C_3)}{2a_2\eta\,(\eta - \mu)} \tag{31}$$

$\bar{E}_0$ then is given by Eq. (32), with C_5 being defined by Eq. (31).

$$\bar{E}_0(\eta, z, s) = \left(\frac{k}{2\eta^2}\frac{c}{s} - \frac{\mu}{\eta}C_5\right)e^{-\eta z} + \frac{k}{2\eta}\frac{c}{s}ze^{-\eta z} + C_5 e^{-\mu z} \tag{32}$$

Some of the transformed stresses, strains, and displacements in the poro-elastic half-space can now be computed using the solutions (30) and (32) for $\bar{S}_0$ and $\bar{E}_0$. $\bar{u}_{z_0}$ may be obtained by transforming the second Eq. (29) of 4.2 and substituting for $\bar{E}_0$ and $\bar{S}_0$.

$$\bar{u}_{z_0} = \left(C_5\mu - \frac{(C_6 - \bar{q}_0^+)}{C_7}\right)e^{-\eta z} - C_5\mu e^{-\mu z} - \left(\frac{kc}{2s} + \eta\frac{(C_6 - \bar{q}_0^+)}{C_7}\right)ze^{-\eta z} \tag{33}$$

The first invariant of the transformed solid strain tensor can also be obtained by making use of the fact that according to Eq. (24) of 4.2, e_{mm} is equal to $\nabla^2 E$.

$$\bar{e}_{mm_0}(\eta,z,s) = \frac{s}{c} C_5 e^{-\mu z} - \frac{c}{s} k e^{-\eta z} \tag{34}$$

The quantity $\bar{p}_0$ can also be obtained by using Eq. (28) of 4.2.

$$\bar{p}_0(\eta,z,s) = \frac{2a_2}{(1+a_1)}\left(-\frac{\eta(C_6-\bar{q}_0^+)e^{-\eta z}}{C_7} + \frac{(a_3+2a_2)\bar{e}_{mm_0}(\eta,z,s)}{2a_2}\right) \tag{35}$$

Using the constitutive equations (1) of 7.1 it follows that the transform of the first invariant of the solid partial stress tensor is related to $\bar{e}_{mm_0}$ and $\bar{p}_0$ as shown in Eq. (36):

$$\bar{\sigma}_{mm_0}(\eta,z,s) = -3a_1\bar{p}_0 + (2a_2+3a_3)\bar{e}_{mm_0} \tag{36}$$

The transform of the vertical partial solid stress in the half-space can be obtained using the second Eq. (2) of 7.1.

$$\bar{\sigma}_{zz_0}(\eta,z,s) = \frac{(a_3-2a_1a_2)}{(1+a_1)}\bar{e}_{mm_0} + 2a_2\left[\left(-\frac{kc}{2s}-\mu\eta C_5 + \frac{a_1}{(1+a_1)}\eta\frac{(C_6-\bar{q}_0^+)}{C_7}\right)e^{-\eta z} + \left(\frac{\eta kc}{2s}+\eta^2\frac{(C_6-\bar{q}_0^+)}{C_7}\right)z e^{-\eta z} + \mu^2 C_5 e^{-\mu z}\right] \tag{37}$$

The other unknowns are more difficult to obtain because their expressions include partial derivatives of $\bar{E}$ and $\bar{S}$, with respect to the variable r. The derivation of explicit expressions for these other quantities is given in 8.3.

8.3 Determining σ_{rz}, σ_{rr}, $\sigma_{\theta\theta}$, and u_r .

σ_{rz} is defined by Eq. (38).

$$\sigma_{rz} = 2a_2\left(\frac{\partial^2 E}{\partial r\partial z} + z\frac{\partial^2 S}{\partial r\partial z}\right) \tag{38}$$

which is one of the constitutive equations from 7.1. Forming the Laplace, and then the first order Hankel transform of this equation results in Eq. (39).

$$\bar{\sigma}_{rz_1} = -2a_2\left(\eta\frac{d\bar{E}_0}{dz} + z\eta\frac{d\bar{S}_0}{dz}\right) \tag{39}$$

By Eq. (39) it can be seen that $\bar{\sigma}_{rz_1}$ can be expressed in terms of $\bar{E}_0$ and $\bar{S}_0$. Equations (30) and (32) of 8.2 provide the necessary expressions of $\bar{S}_0$ and $\bar{E}_0$ that are needed for computing the derivatives which are given by Eq. (40).

$$\frac{d\bar{S}_0}{dz} = -\frac{\eta(C_6-\bar{q}_0^+)\,e^{-\eta z}}{C_7}$$

$$\frac{d\bar{E}_0}{dz} = +\mu C_5\, e^{-\eta z} - \frac{k}{2}\,\frac{c}{s}\; z\, e^{-\eta z} - \mu C_5\, e^{-\mu z} \tag{40}$$

Finally, σ_{rz} can be obtained by performing inversion (41):

$$\sigma_{rz} = L^{-1}\left[\int_0^\infty -2a_2\left(\eta\frac{d\bar{E}_0}{dz} + \eta z\frac{d\bar{S}_0}{dz}\right)\eta\; J_1(\eta r)\,d\eta\right] \tag{41}$$

where L^{-1} indicates the inverse Laplace transformation is to be performed

u_r can be obtained in a similar matter.

$$u_r = \frac{\partial E}{\partial r} + z\frac{\partial S}{\partial r} \tag{42}$$

Equation (42) can be expressed in terms of known transformed quantities as shown in Eq. (43).

$$\bar{u}_{r_1} = -\eta\bar{E}_0 - z\eta\bar{S}_0 \tag{43}$$

The necessary inversion process is indicated in Eq. (44).

$$u_r = L^{-1}\left[-\int_0^\infty (\eta\bar{E}_0 + z\eta\bar{S}_0)\;\eta J_1(\eta r)\,d\eta\right] \tag{44}$$

In order to derive an expression for σ_{rr} that can be computed from $\bar{E}_0$ and $\bar{S}_0$, consider Eq. (45):

$$\frac{\partial}{\partial r}(\sigma_{rr}-p) + \frac{(\sigma_{rr}-\sigma_{\theta\theta})}{r} + \frac{\partial\sigma_{rz}}{\partial z} = 0 \tag{45}$$

which is the first equilibrium equation, Eq. (1) of 4.1 expressed in cylindrical coordinates, using physical components. Multiplying Eq. (45) by r^2 and rearranging terms results in Eq. (46):

$$\frac{\partial}{\partial r}\, r^2(\sigma_{rr}-p) = r(\sigma_{rr}-2p+\sigma_{\theta\theta}) - r^2\frac{\partial\sigma_{rz}}{\partial z} \tag{46}$$

Now if an expression can be obtained for $\bar{\sigma}_{rr_0} - 2\bar{p}_0 + \bar{\sigma}_{\theta\theta_0}$ in terms of $\bar{E}_0$ and $\bar{S}_0$, then Eq. (46) can be integrated with respect to r to obtain the desired expression for σ_{rr}. Such an expression follows by the definition of σ_{mm}.

$$\sigma_{rr} - 2p + \sigma_{\theta\theta} = +\sigma_{mm} - \sigma_{zz} - 2p \tag{47}$$

Transforming Eq. (47) yields the desired expression, Eq. (48):

$$\bar{\sigma}_{rr_0} - 2\bar{p}_0 + \bar{\sigma}_{\theta\theta_0} = \bar{\sigma}_{mm_0} - \bar{\sigma}_{zz_0} - 2\bar{p}_0 \tag{48}$$

The transforms appearing on the right of Eq. (48) all have been determined previously and are given by Eqs. (36), (37), and (35), respectively.

Returning to Eq. (46), it can be seen that the quantity $\frac{\partial}{\partial r}(r^2(\sigma_{rr}-p))$ can be obtained from the inversion formula (49):

$$\frac{\partial(r^2(\sigma_{rr}-p))}{\partial r}=r\int_0^\infty L^{-1}\left[\bar{\sigma}_{rr_0}-2\bar{p}_0+\bar{\sigma}_{\theta\theta_0}\right]\eta J_0(\eta r)\,d\eta$$
$$-r^2\int_0^\infty L^{-1}\left[\frac{d\bar{\sigma}_{rz_1}}{dz}\right]\eta J_1(\eta r)\,d\eta \qquad (49)$$

By making use of Eqs. (41) and (48), integrating both sides of Eq. (49) with respect to r and interchanging the order of integration, Eq. (50) :

$$r^2(\sigma_{rr}-p)=\int_0^\infty\int_0^r r\,L^{-1}\left[\bar{\sigma}_{rr_0}-2\bar{p}_0+\bar{\sigma}_{\theta\theta_0}\right]\,dr\,d\eta$$
$$-\int_0^\infty\int_0^r r^2 L^{-1}\left[\frac{d\bar{\sigma}_{rz_1}}{dz}\right]\eta J_1(\eta r)\,dr\,d\eta \qquad (50)$$

is obtained. Making use of the integrals, Eq. (51):

$$\int_0^r r J_0(\eta r)\,dr=\frac{r}{\eta}J_1(\eta r)$$
$$\int_0^r r^2 J_1(\eta r)\,dr=\frac{2r}{\eta^2}J_1(\eta r)-\frac{r^2}{\eta}J_0(\eta r) \qquad (51)$$

results in Eq. (52):

$$\sigma_{rr}=p+L^{-1}\left[\int_0^\infty(\sigma_{rr_0}-2\bar{p}_0+\sigma_{\theta\theta_0})\frac{1}{r}J_1(\eta r)\,d\eta\right.$$
$$\left.-\int_0^\infty\frac{d\bar{\sigma}_{rz_1}}{dz}\left(\frac{2}{\eta r}J_1(\eta r)-J_0(\eta r)\right)d\eta\right] \qquad r>0 \qquad (52)$$

where $\bar{\sigma}_{rr_0}-2\bar{p}_0+\bar{\sigma}_{\theta\theta_0}$ is defined by Eq. (48) and $\frac{d\bar{\sigma}_{rz_1}}{dz}$ may be obtained by differentiating Eq. (39) as shown in Eq. (53):

$$\frac{d\bar{\sigma}_{rz_1}}{dz}=-2a_2\eta\left[\left(\frac{-kc}{2s}+\frac{\eta kzc}{2s}-\mu\eta C_5\right)e^{-\eta z}\right.$$
$$\left.+\mu^2C_5e^{-\mu z}+z\eta^2\frac{(C_6-\bar{q}_0^+)}{C_7}e^{-\eta z}-\eta\frac{(C_6-\bar{q}_0^+)}{C_7}e^{-\eta z}\right] \qquad (53)$$

Since p can be computed from Eq. (35), Eq. (52) is fully defined and can be used to compute σ_{rr} . Also, once σ_{rr} is computed, then $\sigma_{\theta\theta}$ can be computed using σ_{mm}, σ_{rr} , and σ_{zz} as shown in Eq. (54):

$$\sigma_{\theta\theta} = \sigma_{mm} - \sigma_{rr} - \sigma_{zz} \tag{54}$$

All of the unknown dependent variables have now been expressed in terms of the known transformed displacement generating functions. However, the doubly transformed solution images that have been presented must be inverted in order to provide a meaningful solution of the physical problem being studied. Equations (55)

$$\bar{g}(r,s) = \int_0^\infty \eta\, \bar{g}_\nu(\eta,s)\, J_\nu(\eta r)\, d\eta \tag{55}$$

and (56)

$$g(r,t) = L^{-1}(\bar{g}(r,s)) \tag{56}$$

indicate symbolically the operations that will provide the physical solution. In Eq. (55), $\bar{g}_\nu(\eta s)$ denotes a typical solution image such as $\bar{q}_0^+$. For the ν order Hankel inversion (55), the Laplace variable s is considered constant, then in Eq. (56) s is allowed to vary. The actual inversion algorithm to be used in this study is a numerical approximation of the inversion process indicated by Eqs. (55) and (56). The numerical inversion technique is presented in Chapter IX.

IX

INVERSION OF LaPLACE-HANKEL TRANSFORMS

9.1 General Algorithm

The Laplace-Hankel transformed solutions that were obtained in Chapter VIII have been programmed so that the transforms can be evaluated for given values of η and s using a computer. The general procedure that is used to obtain the desired physical solutions from the transform images is discussed in this section.

Values of the Laplace transform of the solution are obtained at discrete points along the positive real axis in the transformed plane by performing the Hankel inverse transformation with s held fixed. The Hankel inversion is approximated by performing numerical integrations between zeros of the integrand and accumulating the sums until successive intervals yield negligible results. The integrand consists of the product of the double transform with s fixed and the inversion kernel as shown in the following equation:

$$\bar{g}(r,s) = \int_0^\infty \bar{g}_\nu(\eta,s)\Big|_{s=\ell_m} \eta J_\nu(\eta r)\, d\eta \qquad (1)$$

After the approximate Hankel inversion has been performed, the Laplace transform data are used to construct a Fourier series approximation of the time-dependent solution. The approximate solution obtained converges in the mean to the exact solution. A truncated version of the series expansion is used in the computer program.

There are three sources of error in the transform inversion algorithm. In the approximate Hankel inversion, errors occur through the numerical integration process and using finite limits of integration. In the Laplace inversion approximation, error results because only a finite number of terms in the Fourier series are used. Finally, error results due to round-off as in all computer programs.

In order to investigate the accuracy of the computer program, the iterated Laplace-Hankel transforms of several known functions were inverted using the inversion algorithm. The results obtained agreed with the known solutions to a reasonable degree of accuracy and indicated that the accuracy of the algorithm is adequate. Based on the indirect accuracy checks obtained testing known functions, it is thought that all of the algorithm errors combined are within the accuracy of the physical definition of the problem. That is, the errors incurred in determining mechanical constants and other physical parameters in the problem which determine the uniqueness of the mathematical problem are consistent with those inherent in the transform inversion algorithm.

More detailed information follows in 9.2, concerning the theoretical basis of the Laplace inversion procedure.

9.2 Numerical Inversion of the Laplace Transform

The theoretical formulation of the Laplace inversion algorithm used in this study was developed by Erdelyi [13] . He showed that it is theoretically possible to construct Fourier series approximations of functions, quadradically Riemann integrable on the positive infinite internal of real numbers [8], utilizing only the Laplace transforms of the functions. Erdelyi observed [14, 15] that the sequence of function $e^{-l_m t}$ is linearly dense in the vector space $L_2(0,\infty)$, which is defined as the set of quadradically integrable functions mentioned above, if the sequence l_m consists of real, positive numbers and satisfies Eq. (2).

$$\sum_{i=0}^{\infty} \frac{l_i}{1+(l_i)^2} = \infty \tag{2}$$

He also noted that it is possible to construct an orthonormal sequence of functions ϕ_n that is linearly dense in $L_2(0,\infty)$ by applying the Gram-Schmidt orthonormalization process [16] to the functions $e^{-l_m t}$ if the l_m elements satisfy Eq. (3).

$$l_m \neq l_n \quad \text{if} \quad m \neq n \tag{3}$$

Orthonormalization of $e^{-l_m t}$ yields Eq. (4):

$$\phi_n(t) = \sum_{m=0}^{n} C_{mn} e^{-l_m t} \tag{4}$$

which defines the ϕ_n functions.

The coefficients C_{mn} in Eq. (4) are defined by Eq. (5):

$$C_{mn} = (2l_n)^{1/2} \prod_{i=0}^{n-1} (l_m + l_i) \Big/ \prod_{\substack{k=0 \\ (k \neq m)}}^{n} (l_m - l_k) \tag{5}$$

where $\prod_{n=i}^{m}$ denotes the product of terms i through m.

It is, therefore, possible to construct a Fourier series representation of any function $f(t)$ quadradically integrable on the interval zero to infinity, as stated in Eq. (6).

$$f(t) \sim \sum_{n=0}^{\infty} (f, \phi_n)\phi_n(t) \equiv \sum_{n=0}^{\infty} \left(\int_0^{\infty} f(\tau)\phi_n(\tau)d\tau \right) \phi_n(t) \tag{6}$$

---

[8] It is shown in Appendix C that nearly all of the transformed solutions obtained in Chapter VIII correspond to time-dependent solutions that have finite steady state values not equal to zero. Such functions are not quadradically integrable on the infinite interval. However, if the steady state portions of the transforms are subtracted off, the resulting transforms correspond to quadradically integrable functions.

The series on the right of Eq. (6) converges to $f(t)$ in the mean square sense as defined in Eq. (7).

$$\lim_{m\to\infty} \int_0^\infty \left| f(t) - \sum_{n=0}^{m} (f,\phi_n)\,\phi_n(t) \right|^2 dt = 0 \tag{7}$$

This means that $f(t)$ is approximated by the series in such a manner that the integral of the square of the approximation error along the t axis vanishes as the number of terms in the series becomes infinite.

The inner product terms (f,ϕ_n) can be expressed in terms of the values of Laplace transform of $f(t)$ at $s=l_m$ as shown in Eq. (8):

$$\int_0^\infty f(\tau)\phi_n(\tau)\,d\tau = \sum_{m=0}^{n} C_{mn} \int_0^\infty f(\tau) e^{-l_m \tau} d\tau = \sum_{m=0}^{n} C_{mn}\, \bar{g}(l_m) \tag{8}$$

Substituting Eq. (8) in Eq. (6) yields an expression for $f(t)$ which can be computed if $\bar{g}(l_m)$ are known.

$$f(t) \sim \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} C_{mn}\, \bar{g}(l_m) \sum_{m=0}^{n} C_{mn}\, e^{-l_m t} \right) \tag{9}$$

Series expression (9) provides an inversion formula for the Laplace transformation which utilizes knowledge of the transform $\bar{g}(s)$ at the discrete points $s=l_m$. Truncating the series (9) to a finite number of terms defines an algorithm which may be used to approximate $f(t)$ if $\bar{g}(s)$ is known.

9.3 Improving Convergence

It was found in applying the algorithm to the plate on half-space problem transforms that the partial sums of the Fourier series approximation of the solutions in the time domain converge in an oscillatory manner at small values of time. In order to improve the convergence, Fejer summing was performed as defined by Eq. (10) :

$$\text{Fejer Sum}_N = \left(\sum_{n=1}^{N} S_n\right) / N \tag{10}$$

where S_n denotes the nth partial sum of the Fourier series. This summation procedure is known to improve the convergence of Fourier sine-cosine series [17] and was formally applied to the partial sums of the generalized Fourier series as a numerical experiment. Figure 5 shows a comparison between the convergence achieved in a typical problem as a function of the number of terms in the series by both summation methods. It can be seen that in the problem shown, the Fejer summing procedure markedly improved the convergence. Although no proof has been presented concerning the generality of this result, from the numerical results obtained thus far, the Fejer summing procedure appears to be a time saving device that provides correct results.

Another technique which yields improved convergence consists of computing the mean value of the n and $n-1$ partial sums. Results obtained using this procedure are shown in Figure 6. This particular procedure has been incorporated into the computer program.

9.4 Selecting ℓ_m Points

In addition to the requirements stated in 9.1, Erdelyi [13] suggested that the ℓ_m sequence should be a "base" for the Laplace transformation.[9] This means that if the Laplace transformed function is approximated by some new function and the difference between the values of the function and its approximation vanish at all the "base" points, then the difference between the two functions will vanish identically at all points indicating that the two functions are equivalent. Further, by the uniqueness of the Laplace transformation (18) the inverse transform of the approximating function can differ from that of the

[9] Any sequence of points that has a finite limit point is a base for the Laplace transformation [13].

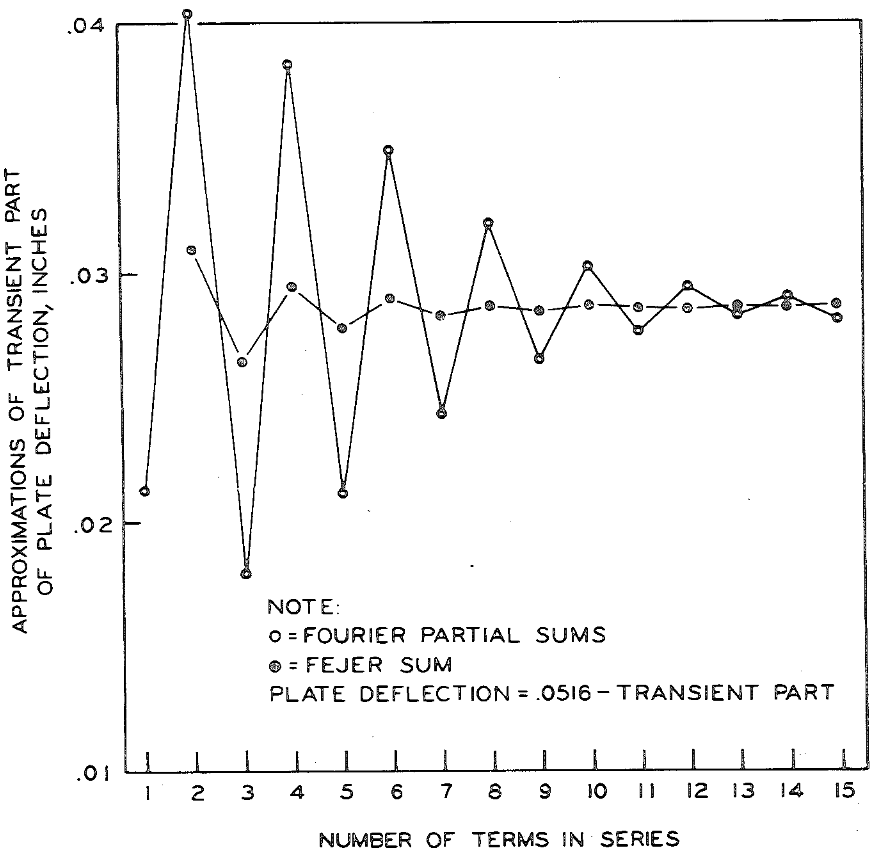


Figure 5. Convergence comparison, Fourier partial sums, and Fejer sums.

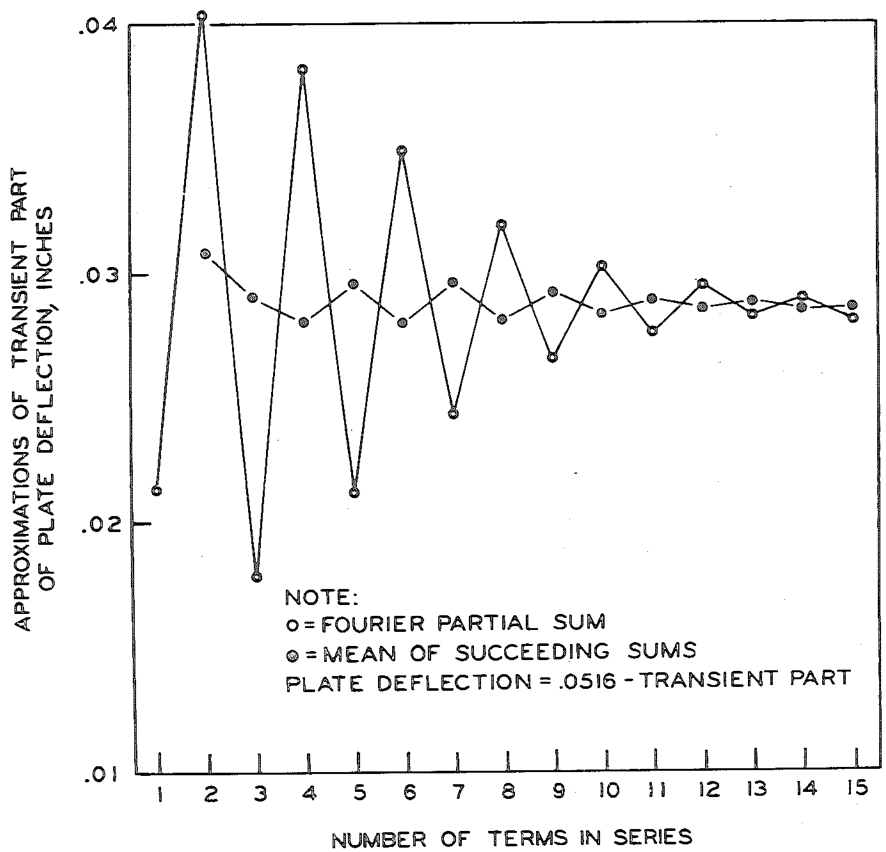


Figure 6. Convergence comparison, Fourier partial sums, and mean of succeeding sums.

original function only by a null function which is of no consequence:

$$\int_0^t \text{Null Function } (\tau)\, d\tau = 0\text{, for every positive } t \tag{11}$$

The practicality of placing this additional restriction on the choice of the ℓ_m sequence becomes apparent if the Laplace transformation is applied to both sides of Eq. (9). Using Schwarz's inequality [16], it can be shown that such a transformation will yield Eq. (12):

$$\bar{g}(s) = \sum_{n=0}^{\infty} \left(\sum_{m=0}^{\infty} C_{mn}\, \bar{g}(\ell_m) \right) \left(\sum_{m=0}^{n} C_{mn} (s+\ell_m)^{-1} \right) \tag{12}$$

where the series shown converges to the transform $\bar{g}(s)$ pointwise as defined by Eq. (13).

$$\lim_{n \to \infty} \left| \bar{g}(s) - \sum_{n=0}^{\infty} (f, \phi_n) \int_0^{\infty} \phi_n(t)\, e^{-st}\, dt \right| = 0 \tag{13}$$

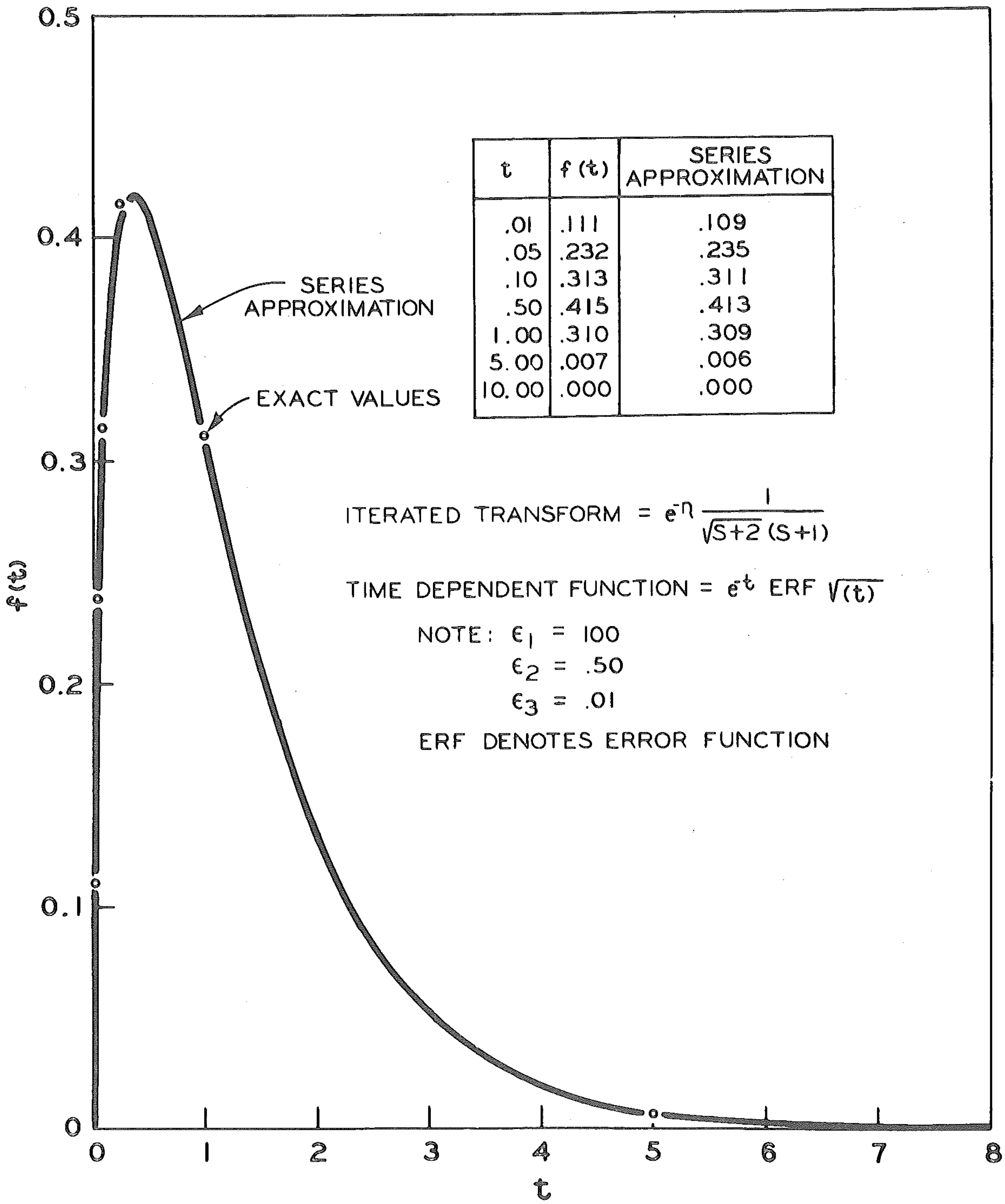
That is, the error of approximation at each point s vanishes as the number of terms in the series become infinite.

Conversely if the ℓ_m are restricted so as to form a base for the Laplace transformation, then Eq. (14):

$$\lim_{n \to \infty} \left| \bar{g}(\ell_m) - \sum_{n=0}^{\infty} (f, \phi_n) \int_0^{\infty} \phi_n(t)\, e^{-\ell_m t}\, dt \right| = 0 \tag{14}$$

implies that the series approximation (12) will converge pointwise to the Laplace transform of the unknown time-dependent solution. Therefore, since it is known that only a finite number of ℓ_m points can be used in numerically constructing the Fourier series approximation (9) of $f(t)$ it seems logical to select the ℓ_m such that (14) is approximately satisfied. This can be done by requiring that the ℓ_m be part of a sequence that is a base for the Laplace transformation.

Several ℓ_m sequences were experimented with in developing the algorithm for inverting the transformed solutions to the plate on half-space problem. All of the sequences tested satisfied the requirements set forth by Erdelyi, but it was found that numerical errors develop in the inversion process if the truncated



t	f (t)	SERIES APPROXIMATION
.01	.111	.109
.05	.232	.235
.10	.313	.311
.50	.415	.413
1.00	.310	.309
5.00	.007	.006
10.00	.000	.000

Figure 7. Numerical inversion results, test function.

sequences that are used do not satisfy certain additional conditions.

Generally, it was found that the maximum and minimum magnitudes of the ℓ_m points must be limited. It was also discovered that the convergence of the ℓ_m sequence at its limit point must not be too rapid. That is, the ℓ_m points must be spread out over the domain defined by the maximum and minimum allowable values, rather than clustered near the limit point.

The ℓ_m sequence that was selected for use in the computer program is defined as follows:

$$\begin{aligned} \ell_0 &= \epsilon_1 \\ \ell_n &= \ell_{n-1} \times \epsilon_2 + \epsilon_3 \end{aligned} \tag{15}$$

The parameters ϵ_1, ϵ_2 and ϵ_3 are determined by examining the Laplace transform image. ϵ_1 and ϵ_2 are selected to bracket the portion of the transform image that maps into the transient part of the solution in the time domain. ϵ_1 defines the maximum ℓ_m value and ϵ_2 and ϵ_3 define the rate of convergence of the sequence and the magnitude of the smallest ℓ_m value. Generally, ϵ_1 may be set equal to 100, $\epsilon_2 = .5$ and $\epsilon_3 = .01$. Figure 7 illustrates the results that were obtained applying the inversion algorithm to the iterated Laplace-Hankel transform of a known time-dependent function using these values of ϵ_1, ϵ_2 and ϵ_3.

X

EXAMPLE PROBLEM

The following numerical example is presented to illustrate the use of the computer program and to provide a test problem for potential program users. Only the results of the problem analysis are presented here. Details concerning program input and output are given in the Appendix along with the program listing.

A hypothetical pavement structure having the geometric and physical properties indicated in Figure 8 is assumed to be subjected to the loading shown at time zero. The foundation constants that are shown were approximated for a saturated, well-graded sand and gravel mixture using the techniques discussed in Appendix B-1. The stress relaxation function $\Phi(t)$ that is used in this example to describe the plate material is based on stress relaxation tests that were performed on sand asphalt mixtures by Moavenzadeh and Soussou [19]. Details concerning the stress relaxation data are given in Appendix B-2.

The results of the computer analysis are as follows: the reaction of the foundation directly under the center of the load is approximately equal to 32 psi at time zero and approaches a steady state value of 40 psi rapidly. At a distance of one load radius, or 10 in. from the origin, the foundation reaction is 16 psi at $t = 0$ and 19 psi at steady state. The partial fluid stress at the origin varies from 3.7 psi, or 11 percent of foundation reaction, at $t=0$ to zero value

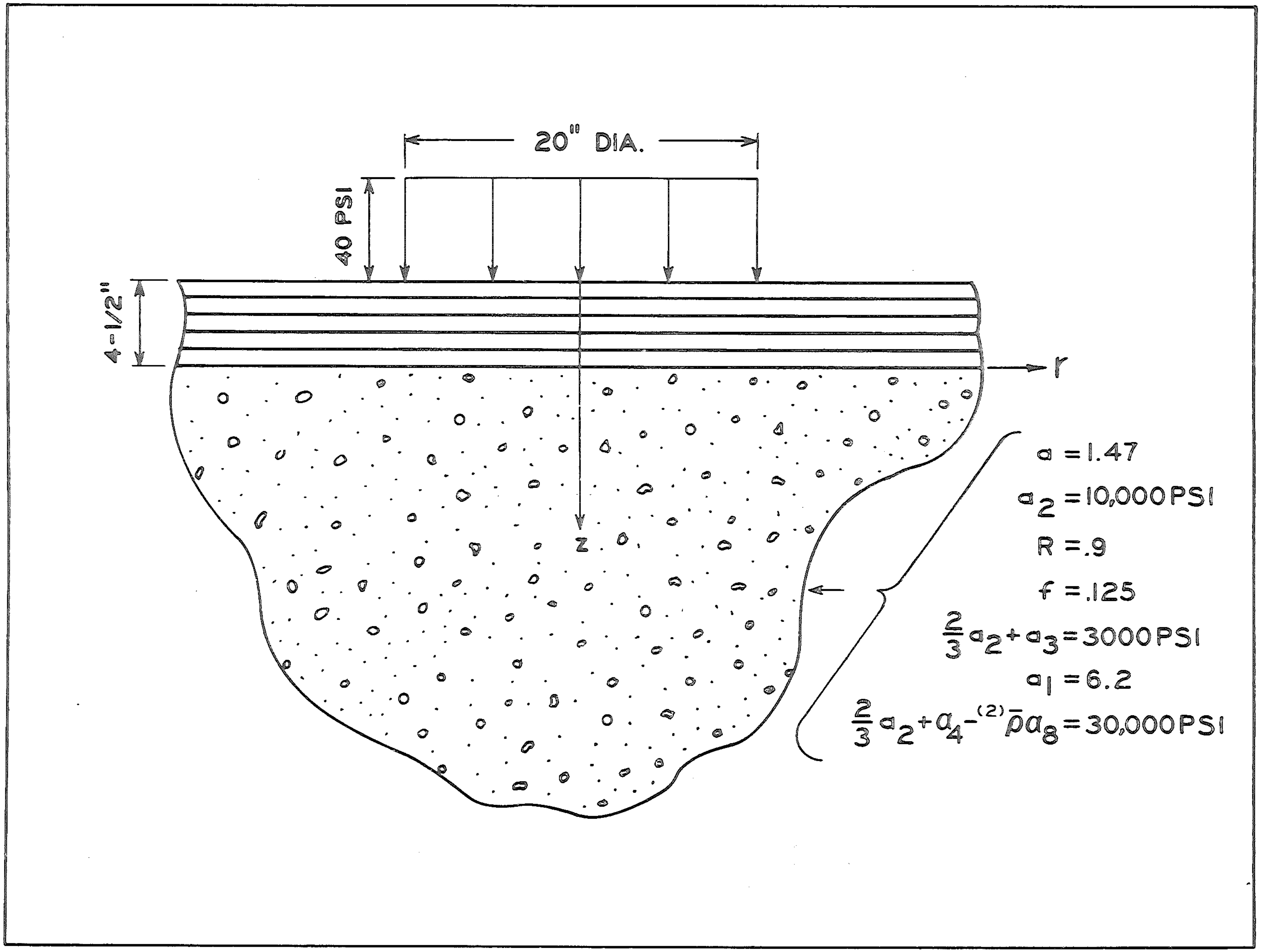


Figure 8. Example pavement structure.

in steady state as shown in Figure 9. The variation of fluid pressure with depth is shown in Figure 10. The plate deflection at the origin is shown as a function of time in Figure 11, having an initial value of about 0.02 in. and a steady state value of 0.05 in. The plate deflection at 10 in. from the origin varies from an initial value of 0.015 in. to a steady state value of 0.03 in. as shown in Figure 12.

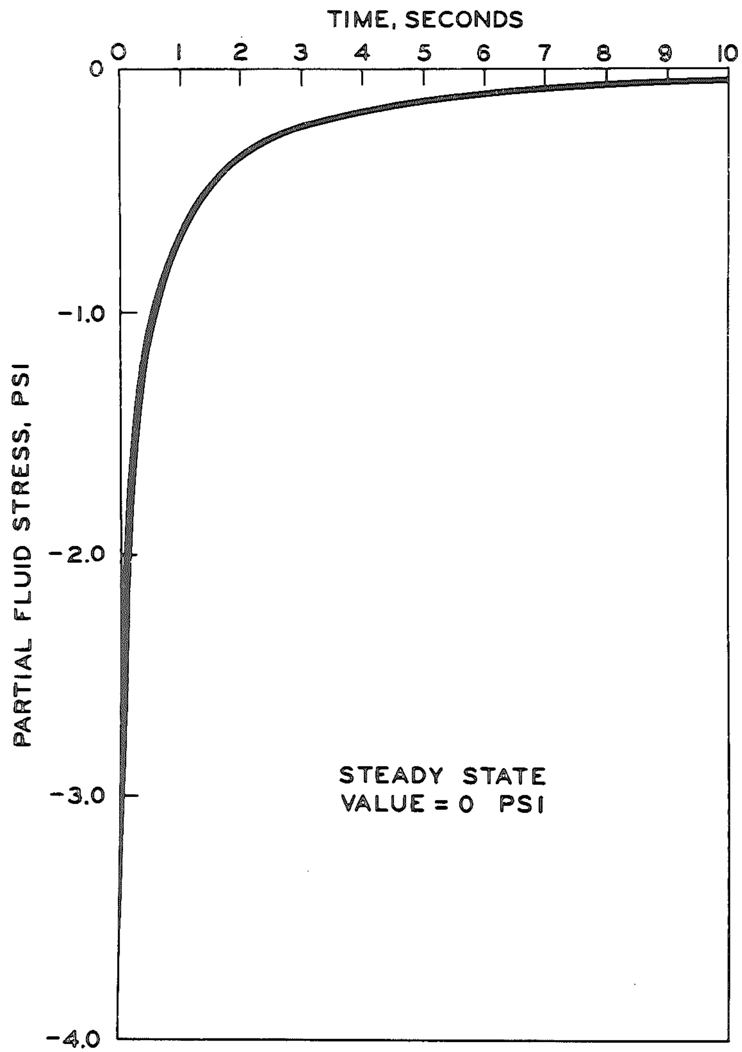


Figure 9. Partial fluid stress at origin versus time.

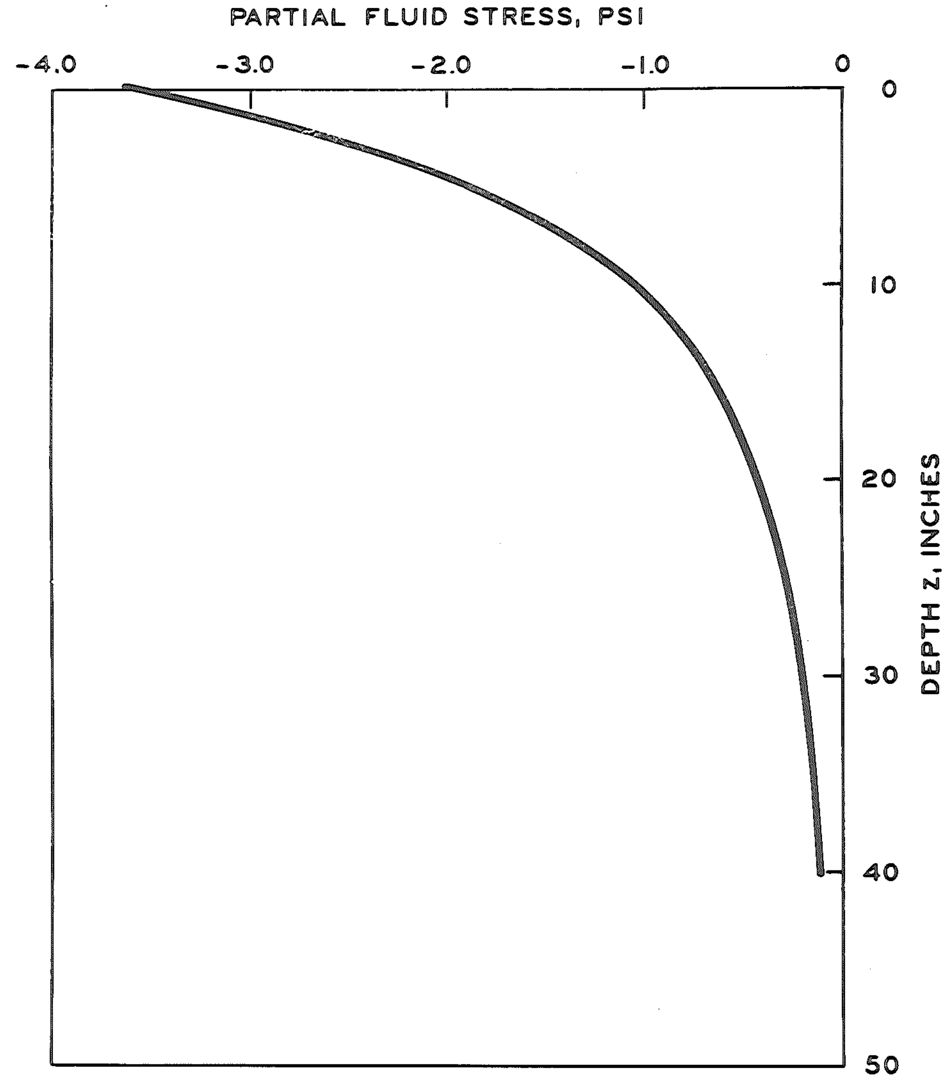


Figure 10. Partial fluid stress variation with depth at centerline of load, time = zero.

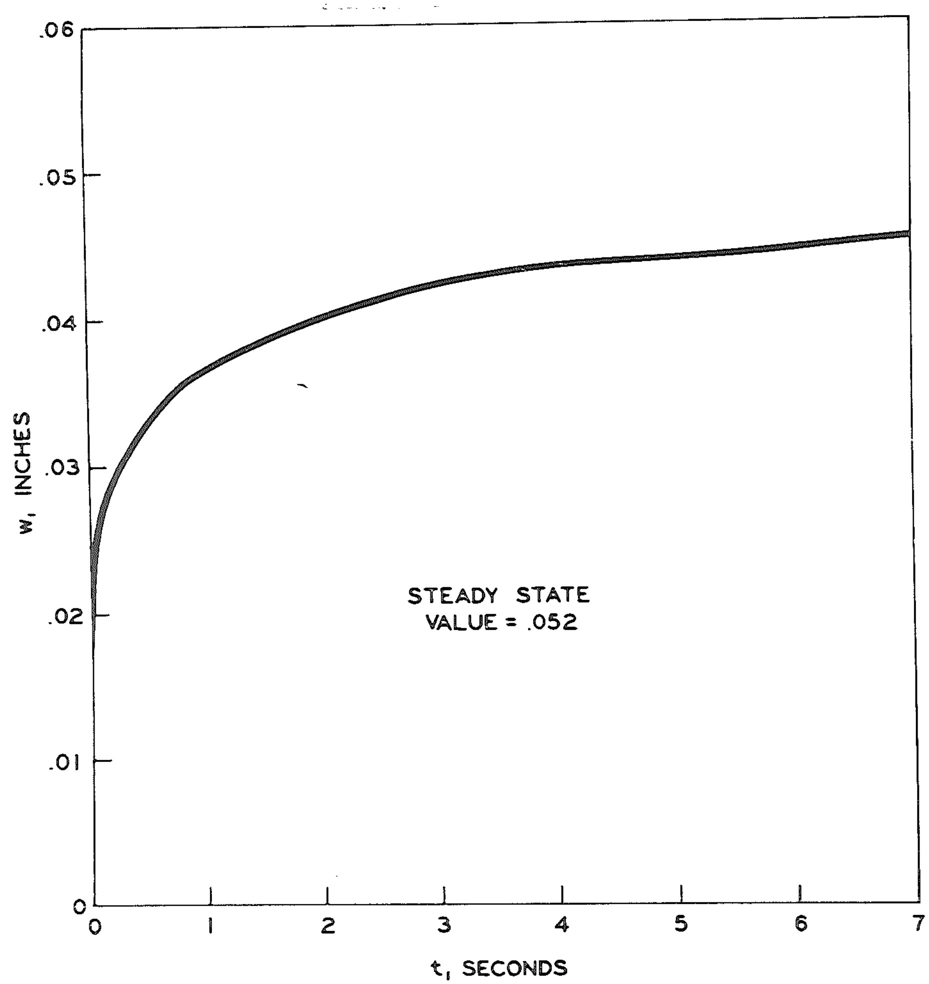


Figure 11. Surface deflection at origin versus time.

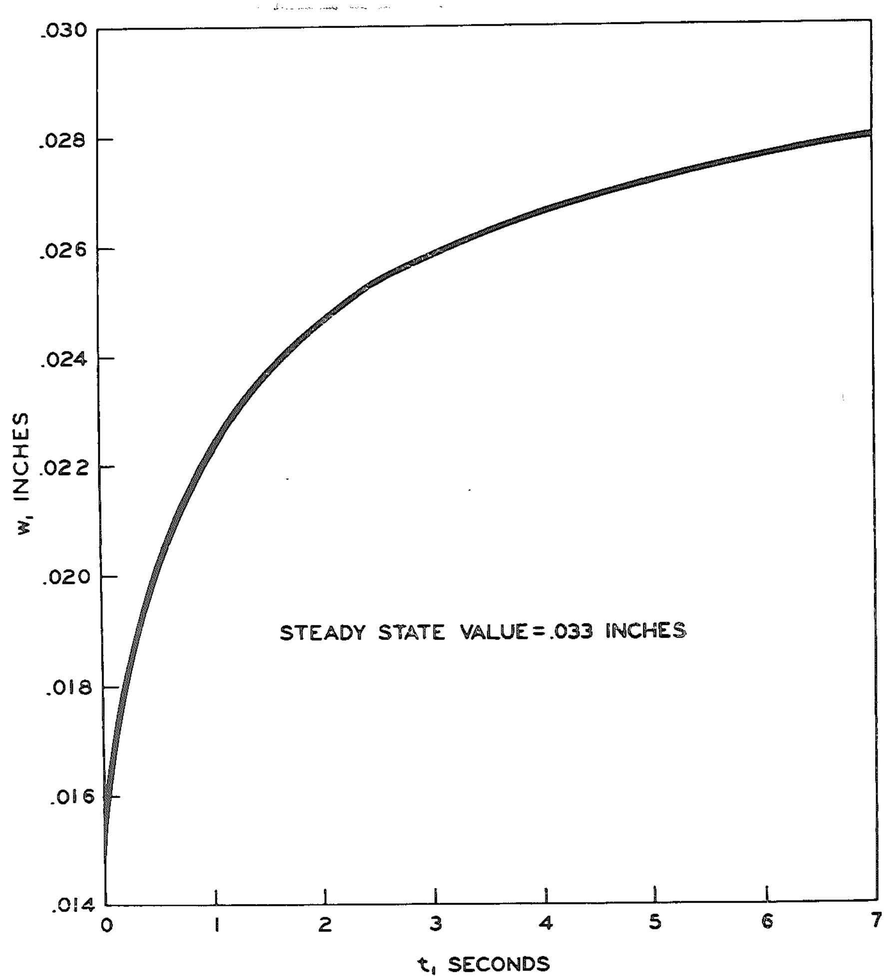


Figure 12. Surface deflection at radial distance of 10 inches from origin versus time.

XI

CONCLUSION

11.1 Summary

In this study a method has been developed for analyzing the deformation response to load of a flexible pavement model consisting of a viscoelastic plate supported on a poro-elastic foundation. The three stated objectives in the proposal were accomplished. The mathematical model allows for an initial volume change to take place when the load is applied in order to account for the fact that the solid particles in granular soil material may be compressible. This allowance for initial compressibility has not been allowed for in previous foundation settlement theories.

The model response to applied load was formulated as an initial-boundary value problem. The technique used to solve the problem consisted of reducing the partial differential equations of the problem into ordinary differential equations using transform methods. Then the transformed problem was solved yielding iterated Laplace-Hankel transformed images of the desired solutions. A numerical technique was then developed to invert the transforms and obtain the desired physical solutions, the stresses and displacements in the model. The results of the analysis have been recorded in a computer program which is included in this report. Using the program, it is possible to compute plate deflections, foundation to plate reactions, fluid pressures, and eight other stress or displacement quantities as functions of time. The program is designed to be used by others, and instructions concerning its use are given in Appendix A. It is noted that the portion of the program that was developed for inverting Laplace-Hankel transforms is applicable to other problems involving the inversion of either iterated, or single transforms of the types mentioned here.

11.2 Further Study

The most difficult problem that is encountered when one attempts to estimate stresses and displacements in an actual pavement structure utilzing the developed computer program is that of determining the values of the material constants. To begin with,stress relaxation data are required to describe the plate material, and uniaxial stress relaxation tests at constant temperature are required to determine the needed data. The data currently available in the literature need to be extended. There are also seven poro-elastic constants which are used in the analysis. Suggested tests for determining these constants are presented in Appendix B-1. Several tests were made to determine the poro-elastic constants of a typical base course material for use in the sample problem analysis given in Chapter X. The tests were performed using a triaxial soil testing device. From the limited experience gained in performing these tests it appears that although the tests are difficult to perform, such testing is feasible. The poro-elastic properties of base course and subbase materials need to be investigated extensively.

After the poro-elastic and viscoelastic material properties of highway pavement structure materials have been investigated adequately, it will be feasible to check the accuracy of the flexible pavement model. Such verification should include plate load testing of real pavement sections having known material properties. If the model proves accurate for simple pavement structures consisting of a single layer of bituminous material resting on a single foundation material it would then be useful to extend the model to account for layers of different porous media occurring in the foundation. Another phenomenon that will require investigation is the effect of partial saturation on the foundation. If isothermal deformation is assumed, it may be possible to allow for this in the mathematical model by treating the pore fluid as compressible [1].

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REFERENCES

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APPENDIX

APPENDIX A

COMPUTER PROGRAM LISTING AND USER GUIDE

What the Program Does

The program determines the magnitude of any stress or displacement component at any appropriate position in the plate on half-space flexible pavement model as a function of time. The user must specify which unknown stress or displacement function is required, loading conditions, the magnitudes of material constants, and the position where the solution is to be determined.

Using the input information specified, and function sub-routines, the program constructs discrete values of the Laplace-Hankel transformed image of the desired solution. These data are integrated numerically for fixed values of the Laplace variable to yield discrete values of the Laplace transformed image of the solution. (This numerical integration process approximates the exact Hankel transform inversion operator.) The program then uses the discrete data to numerically invert the Laplace transformed solution to obtain the desired physical solution. The Laplace inversion process that is used consists of constructing a Fourier series approximation of the time-dependent solution using the discrete Laplace transform data. The Fourier series approximates the exact solution in the mean square sense, and its Laplace transform approximates the Laplace transform of the solution pointwise.

Execution, or process time, on this program varies from two to ten minutes for one set of data.

The Program ALGORITHM

The algorithm begins in the main program where SET UP is called. In SET UP the problem data is read in and then printed out. The zeroes of the appropriate Hankel kernel are also computed in SET UP and stored in the array ATJ ( ). Control then returns to the main program and the steady state and initial values are obtained and stored in STEADY and F(1). To do this the main program calls HANKEL setting GCODE equal to 2 and 3 and s equal to 1. Utilizing the ATJ ( ), HANKEL calls SMPSM repeatedly where numerical integration is performed on the inverse Laplace transform of the desired solutions between successive ATJ ( ) points. Within SMPSN the function G is called which in turn calls one of the G 1 through G 11 functions in order to evaluate the integral as required for numerical integration. The integral results between successive ATJ ( ) are accumulated in HANKEL until additional integrations contribute less then DEL x 100% of the accumulated results.

The main program then determines the transient part of the solution. The L ( ) sequence and C (,) array are computed in the main program and are used to construct a set of orthonormal functions using the Gram-Schmidt process. The Laplace transform of the desired solution is then determined at each of the L ( ) points and is stored in GINTP ( ). This is done by setting GCODE equal to 1 and s equal to successive values of L ( ) and calling function HANKEL where SMPSN is called as was done in determining STEADY and F(1).

The main program utilizes C (,), L ( ), and GINTP ( ) to construct a truncated Fourier series approximation of the transient part of the solution for each specified value of time, T ( ). The transient solution is stored in F(2) through F (KK) and then combined with STEADY to yield a NN term series approximation of the problem solution, Q ( ) which is then output. Mean sum approximations, which are defined as the sum of the NN and NN-1 approximations of Q ( ) divided by two, are also computed and are stored in QMEAN ( ). Solution approximation results are output consisting of F ( ), Q ( ) and QMEAN ( ) for each NN series approximation. This process is repeated NN equal I through NNN and successive series approximations are compared. The process is terminated before NNN is reached if a succeeding approximation differs from the previous by less than SUMDEL x 100%. The final QMEAN ( ) approximation is output as the solution along with the times T ( ).

Card Input Description

Order of Precedence as given:

Variable Names	Format	Optional
MG, OPT1, OPT2, OPT3	4I2	No
KK, (T (I), I = 2, KK)	I2,8X,7D10.5/(8D10.5)	Yes
DEL, BJDEL, IMAX, NJO, SUMDEL	F10.5, F16.10, 2I2, F10.5	Yes
L1,ACCPT, FOCUS, NNN	3D6.3, I2	Yes
Z, RR, H, BE, Q	5F10.5	No
FF, R, BULKO, BULK, GAMA, DARCY, A2, A1	2F10.5, 6E10.5	No
RO, NB	E10.4, I2	No
B(I), I = 1, NB	8E10.4/	No
BB(I), I = 1, NB	8E10.4/	No

Variable Description

OPT1, OPT2, OPT3	determine which optional input cards are being utilized. Setting OPT1, OPT2, and OPT3 equal to zero indicates that no optional data cards are being used. Setting OPT1 equal to one indicates that the time array is to be specified. Setting OPT2 equal to one indicates that DEL, BJDEL, IMAX, NJO and SUMDEL are specified. Setting OPT3 equal to one indicates L1, FOCUS, ACCPT, and NNN are to be specified.
MG	determines which stress or displacement component is to be computed. When it has a value = 1, the foundation reaction stress at the plate boundary is to be computed. = 2, the plate deflection is to be computed at some point. = 3, implies the vertical displacement component at some point in the foundation is to be computed. = 4, implies the vertical stress component is to be computed. = 5, implies the first invariant of the solid strain tensor is to be computed. = 6, implies the partial fluid stress is to be computed. = 7, implies the first invariant of the solid partial stress tensor is to be computed. = 8, implies the solid radial displacement component is to be computed.

=9, implies the shear stress, σ_{zr} is to be computed.

=10, implies the radial stress component of the solid partial stress tensor is to be calculated.

=11, implies the tangential component, $\sigma_{\theta\theta}$, of the solid partial stress tensor is to be calculated.

KK — indicates the number of time values at which the solution is desired.

T (I) — the time array, having KK elements, the first of which is automatically specified as zero in the program. Only KK-1 values may be read in as input.

DEL — is a decimal tolerance factor. DEL controls the accuracy of the numerical integration performed in sub-routine function HANKEL, but not the accuracy of the numerical Laplace transform inversion that is performed in the main program.

BJDEL — is a decimal tolerance factor. BJDEL controls the accuracy of the Bessel function values that are computed in sub-routine function BJ (,).

IMAX — limits the number of points used in performing numerical integration over a given interval using sub-routine SMPSN.

NJO — limits the maximum number partitions that may be used in sub-routine function HANKEL in performing the numerical Hankel transform inversion process.

SUMDEL — is tolerance factor that controls number of terms used in series approximations of solution.

L1	is largest element L(1) in L ( ) array.
ACCPT	is equal to .5 times the limit point of L ( ) array.
FOCUS	is a constant that controls the convergence of the L ( ) array. It is greater than .1 and less than 1.
NNN	limits the number orthonormal vectors that may be constructed for use in the Fourier series approximation of Laplace transforms in the main program. NNN is the maximum number of terms that will be allowed in the truncated Fourier series.
Z	is the vertical depth to a given point in the model. Z is measured positive downward and is equal to zero at the plate - half-space interface.
RR	is the horizontal radial distance from the axis of the applied load to a point in the model.
H	is the plate thickness. This value is used in computing the bending stiffness of the plate.
BE	is the radius of the loaded circular area.
Q	is the applied load pressure.
FF	is the porosity of the foundation material.
R	is the pore compressibility constant.
BULKO	is the apparent initial bulk modulus of the porous fluid-filled material.
BULK	is the apparent bulk modulus of the porous fluid-filled material in th steady state.
GAMA	is the fluid density of fluid component of the interacting mixture.

DARCY	is the Darcy coefficient of permeability of the porous medium.
A2	is the shear modulus of the solid material.
A1	is the fluid pressure interaction coefficient.
RO	is the initial value of the stress relaxation function of the plate material.
NB	is the number of exponential terms that are to be used to describe the stress relaxation function of the plate material.
B (I)	is an array that contains the exponents of the exponential terms used in relaxation function.
BB (I)	is an array that contains the coefficients of the exponential terms used in the relaxation function.

Sample Input

Problem:

Utilizing the foundation and plate material properties given in this report, determine the plate deflection directly under the loaded area at the origin of the model. The loading pressure is assumed to be 40 psi and the radius of the loaded area, 10 in. The plate is assumed to be 4-1/2 in. thick. The solution to this problem is given in Chapter X.

The required input is as follows:

MG = 2

OPT1 = 0

OPT2 = 0

OPT3 = 0

Z = 0.

RR	=	0.
H	=	4.5
BE	=	10.
Q	=	40.
FF	=	.125
R	=	.9
BULKO	=	30000.
BULK	=	3000.
GAMA	=	.0361
DARCY	=	.196
A1	=	6.2
A2	=	10000.
RO	=	156029.
NB	=	11

B(I) I=2, NB = 5000., 500., 50., 5., .5, .05, .005, .0005, .00005, .000005

BB(I) I=2, NG = 6650, 23200, 41900, 33000, 30100, 12600, 4300, 1440, 836, 403

Note: The option to use no optional cards has been chosen, therefore, certain variables will be automatically specified in the program as follows:

KK	=	20
T (I), I = 1, KK	=	0, .01, .05, .1, .25, .5, 1., 1.5, 2., 2.5, 3.0, 3.5, 4., 4.5, 5., 7., 10., 15., 20., 25.
DEL	=	0.001
BJDEL	=	.001
SUMDEL	=	.05
IMAX	=	8
NJO	=	60

NNN = 14

L(1) = 1.D2

FOCUS = .5

ACCPT = .01

The required input cards appear as shown in Figure 1 .

A complete program listing follows these instructions. Comment cards have been inserted in the listing to explain various parts of the program.

Optional Input Data

Certain input data are optional in that the user need not specify these items unless he desires. The input is set up in this way to minimize the required input but to also insure maximum program flexibility

Special Cases

σ_{rr}, $RR \neq 0$:

In this case it is necessary to submit an extra set of input cards with MG set equal to 12. The reason for this is that in this case it is necessary to perform two separate integrations in computing the Laplace transform of σ_{rr} . One integration is performed on G12, and the other integration is performed on G10.

If $z = RR = 0$ then only one set of data is required with MG set equal to 10.

$\sigma_{\theta\theta}$, $RR \neq 0$:

The program will not solve for $\sigma_{\theta\theta}$ directly. However, $\sigma_{\theta\theta}$ may be obtained by solving for σ_{mm}, σ_{zz} and σ_{rr} separately using the program, and then combining the results according to the following formula.

$$\sigma_{\theta\theta} = \sigma_{mm} - \sigma_{rr} - \sigma_{zz}$$

If $RR = 0 = z$, then G11 may be obtained directly by setting MG equal to 11.

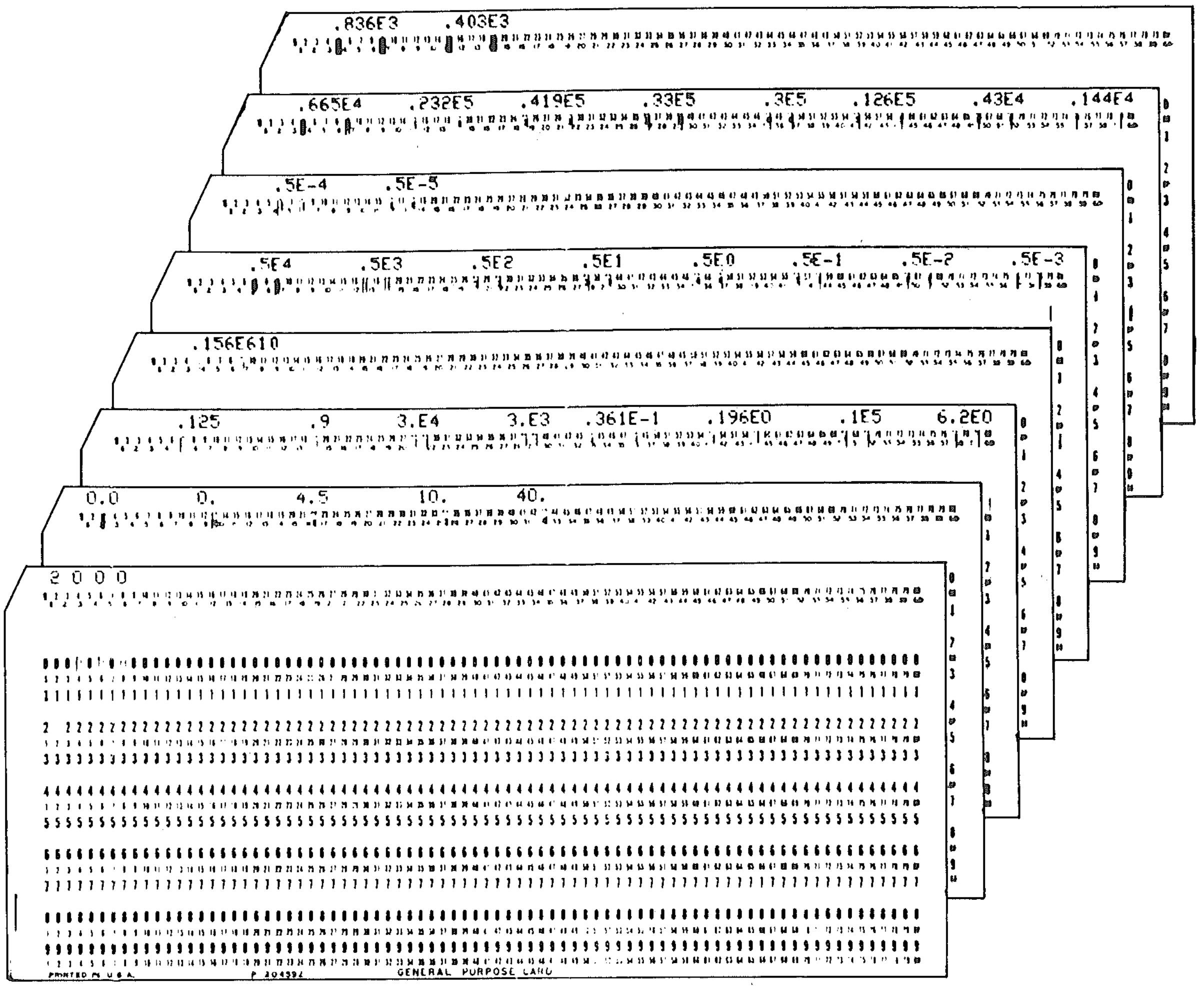

Figure 1 . Typical input cards.

MAIN PROGRAM FOR

FLEXIBLE PAVEMENT MODEL

```
C         BELOW IS THE MAIN PROGRAM FOR THE ANALYSIS OF A
C         LINEAR VISCOELASTIC PLATE SUPPORTED ON A PORO-ELASTIC
C         HALF SPACE AND SUBJECTED TO A UNIFORM CIRCULAR LOAD.
C         REQUIRED SUBROUTINES - SETUP, SMPSN, TIMER
C         REQUIRED SUBROUTINE FUNCTIONS - HANKEL, BJ, G, G1,G2...G12.
C         REQUIRED INPUT ON CARDS - NG, OPT1, OPT2, OPT3
C                                   Z,RR,H,RE,Q
C                                   FF,R,BULK,GAMA,DARCY,A2
C                                   RO, NA
C                                   B(NA)
C                                   BB(NB)
C         OPTIONAL INPUT ON CARDS - KK, T(KK)
C                                   DEL,BJDEL,IMAX,NJO,SUMDEL
C                                   L(1),ACCPT,FOCUS,NNN

      DIMENSION F(100),QB4(100),S3(50),GASP(50),Q(100),GINTP(40)
      DIMENSION REM1( 5),REM2( 5),REM3( 5),REM4( 5),TITLES( 55) -
      DOUBLE PRECISION              L(41),C(40,40),T(101),S(50),
     1 S1(50,40), S2(40),               EXS(100,40)
      DOUBLE PRECISION               AT, L1,     XL,R, TPROD,APROD
      DOUBLE PRECISION DEXP,DABS,XEXS,ACCPT,FOCUS
      COMMON /PART/ATJ(300),NPU,DEL,IMAX
     2       /MODE/ GCODE
     3       /MAIN/NG,NNN,KK,RRMOVE,SUMDEL
     4       /MAINDB/ T,L1, ACCPT,FOCUS
     8       /ERR/NER1,NER2,NER3,NER4
      DIMENSION            QMEAN(100),STORE(100)
      INTEGER P
      DATA(TITLES(I),I=  1, 5 ) / "REACTIVE PRESSURE UNDER PLATE " /
      DATA(TITLES(I),I=  6,10 ) / "        PLATE DEFLECTION      " /
      DATA(TITLES(I),I= 11,15 ) / " SOLID VERTICAL DISPLACEMENT  " /
      DATA(TITLES(I),I= 16, 20) / "SOLID PARTIAL VERTICAL STRESS " /
      DATA(TITLES(I),I= 21, 25) / "       SOLID  DILATATION      " /
      DATA(TITLES(I),I= 26, 30) / "  PARTIAL FLUID STRESS        " /
      DATA(TITLES(I),I= 31, 35) / "1ST INVARIANT-SOLID STRESSES  " /
      DATA(TITLES(I),I= 36, 40) / " SOLID RADIAL DISPLACEMENT    " /
      DATA(TITLES(I),I= 41, 45) / "    SOLID SHEAR STRESS        " /
      DATA(TITLES(I),I= 46, 50) / "    SOLID RADIAL STRESS       " /
      DATA(TITLES(I),I= 51, 55) / "    SOLID TANGENTIAL STRESS   " /
      DATA( REM1(I),I=1, 5)/ "INCREASE NJO BY 2              " /
      DATA( REM2(I),I=1, 5)/ "INCREASE IMAX BY 1             " /
      DATA( REM3(I),I=1, 5)/ "INCREASE    BJDFL              " /
      DATA( REM4(I),I=1, 5)/ "ADJUST OPT3 PARAMETERS         " /
 5000 FORMAT(1H3,7H 012992,20X,23HFLEXIBLE PAVEMENT MODEL)
 5002 FORMAT(1H0,17X,45HVISCOELASTIC PLATE ON PORO-ELASTIC HALF SPACE)
 5005 FORMAT (1H0,10X,24HSTEADY STATE SOLUTION = ,E10.3 )
 5006 FORMAT(1H0,21X,14H** SOLUTION **)
 5013 FORMAT(//)
 5015 FORMAT( / )
 5020 FORMAT(1H ,10X,A6,1H=,I3)
 5040 FORMAT(1H0,25X,5A6)
 5100 FORMAT(1H0, 10D10.3)
 5200 FORMAT (1H0,10X,2HT=,E10.3, 6H  SOL=, E10.3 )
 5900 FORMAT (1H0,49H *** WARNING ***  DESIRED ACCURACY NOT REACHED IN)
 5910 FORMAT (1H ,18X,A6,1X,I4,25H TIMES. SUGGESTED REMEDY=,5A6)
 5920 FORMAT(1H ,1HS)
 5930 FORMAT(1H ,17HLAPLACE TRANSFORM)
 5940 FORMAT(1H ,33HSERIES APPROXIMATION OF TRANSFORM)
 5950 FORMAT(1H ,I3,1X,37HTERM SERIES APPROXIMATION OF SOLUTION)
 5960 FORMAT(1H0,28HPARTIAL SUM - TRANSIENT PART)
 5980 FORMAT(1H0,28HPARTIAL SUM - TOTAL SOLUTION)
 5990 FORMAT(1H0,44HMEAN LAST TWO PARTIAL SUMS - TOTAL SOLUTION )
 5995 FORMAT (1H0,6H TIMES)
      PLACE1 = 6HHANKEL
      PLACE2 = 6HSMPSN
      PLACE3 = 6HBJ
      PLACE4 = 6HMNPROG
   10 CONTINUE
      CALL SETUP
      L(1) =L1
```

```
      RR=RRMOVE
      IF(NG.EQ.10.AND.RR.NE.0.) GO TO 11
      STORF1 = 0.
      STORSS = 0.
      DO 111 I = 1,NNN
      STORE(I) = 0.
  111 CONTINUE
   11 CONTINUE
      NER1 = 0
      NER2 = 0
      NER3 = 0
      NER4 = 0
      IF(NNN.EQ.0) GO TO 9290
C                GET STEADY STATE SOLUTION
      IF(NG.EQ.6) GO TO 5
      GCODE=2.
      IF(NG.EQ.9) GO TO 3
      IF((NG.EQ.12.OR.NG.EQ.10).AND.(RR.NE.0.)) GO TO 3
      STEADY=HANKEL(1.D0)
      GO TO 8
    3 CONTINUE
      GCODE = 1.
      STEADY=HANKEL(1.D-6 )*1.D-6
      GO TO 8
    5 CONTINUE
      STEADY = 0.
    8 CONTINUE
      GCODE=1.
      F(1)=0.0
      GCODE=3.
      F(1) = HANKEL(1.D0)
      IF(NG.EQ.10.AND.RR.NE.0.)F(1)=F(1)+STORF1
   13 CONTINUE
      DO 14 P=2,KK
      QB4(P)=.0D0
   14 CONTINUE
```

```
      GCODE = 1.
      LL = 2
      XL = L(1)
      Q(1) = F(1)
      QMEAN(1) = Q(1)
      S(1) = L(1)
      GINTP(1)=HANKEL(XL)-STEADY/XL
      IF(NG.EQ.10.AND.RR.NE.0.)GINTP(1)=GINTP(1)+STORE(1)
      GASP(1) = GINTP(1)
      DO 15 N=2,NNN
      L(N)  =L(N-1)*FOCUS+ACCPT
   15 CONTINUE
      DO 60 M=1,NNN
      DO 60 N=M,NNN
      IF            (N.EQ.1) GO TO  60
      TPROD = 1.0D0
      DO 30 I=1,M-1
      IF(I.EQ.M) GO TO 30
      TPROD = (L(M)+L(I))/(L(M)-L(I))*TPROD
30    CONTINUE
      IF(N.EQ.M) GOTO 41
      IF(N.GT.M). TPROD=2.0D0*L(M)*TPROD
      IF(N.EQ.M+1) GOTO 40
31    CONTINUE
      DO 40 I=M+1,N-1
      TPROD = TPROD*(L(M)+L(I))/(L(M)-L(I))
40    CONTINUE
      TPROD = TPROD/(L(M)-L(N))
41    CONTINUE
      XL = 2.D0 * L(N)
      C(M,N) = DSQRT(XL)        * TPROD
   60 CONTINUE
      XL = 2.D0* L(1)
      C(1,1) = DSQRT(XL)
      DO 9000 NN=1,NNN
```

```
      NER4=0
      MM = NN
   16 CONTINUE
      IF(NN.EQ.1) GO TO 19
      DO 20 M=LL,NN
   18 CONTINUE
      XL = L(M)
      GINTP(M)=HANKEL(XL)-STEADY/XL
      IF(NG.EQ.10.AND.RR.NE.0.)GINTP(M)=GINTP(M)+STORF(M)
C                       S( ) ARBITRARILY SET = L( )
      S(M) = L(M)
      GASP(M) = GINTP(M)
   20 CONTINUE
   19 CONTINUE
      IF(NG.EQ.12) GO TO 160
      DO  80 N=1,NN
      S2(N) = 0.D0
      DO  80 M=1,N
      S2(N) =(C(M,N)*GINTP(M)                                   )+S2(N)
   80 CONTINUE
C                      REMOVE "GO TO 121" TO OBTAIN SERIES
C                      APPROX OF TRANSFORM
      GO TO 121
      DO 90 P=1,MM
      DO 90 N=1,NN
      S1(P,N)=0.D0
      DO 90 M=1,N
      S1(P,N)=(C(M,N)*1.0/(S(P)+L(M)))+S1(P,N)
   90 CONTINUE
      DO 120 P=1,MM
      S3(P) = 0.D0
      DO 110 N=1,NN
      S3(P)=S1(P,N)+S2(N)+S3(P)
  110 CONTINUE
  120 CONTINUE
      WRITE(3,5000)
      WRITE(3,5920)
      WRITE(3,5100)(S(P),P=1,NN)
      WRITE(3,5930)
      WRITE(3,5100)(GASP(P),P=1,NN)
      WRITE(3,5940)
      WRITE(3,5100)(S3(P),P=1,NN)
  121 CONTINUE
      DO 150 P=2,KK
      F(P) = 0.D0
      DO 130 N=1,NN
      FXS(P,N) = 0.D0
      DO 130 M=1,N
      XEXS = L(M) * T(P)
      IF (XEXS.GT.1.1D2) GO TO 130
      XEXS = -XEXS
      FXS(P,N) = C(M,N) *DEXP( XEXS      ) + FXS(P,N)
  130 CONTINUE
      DO 140 N=1,NN
      F(P) = FXS(P,N) * S2(N) + F(P)
  140 CONTINUE
      IF(    F(P).EQ..0)NER4=1
      IF(NN.GE.2 .AND. QMEAN(P).NE.0.)
     1    ERROR =(((Q  (P)+F(P)+STEADY+STORSS)/2.) - QMEAN(P))/QMEAN(P)
      IF (ABS(ERROR).GE.SUMDEL)NER4=1+NER4
      QB4(P) = Q(P)
      Q(P) = F(P) + STEADY+STORSS
      QMEAN(P) = (QB4(P) + Q(P)) / 2.
  150 CONTINUE
      WRITE (3,5000)
      WRITE(3,5950) NN
      WRITE(3,5995)
      WRITE(3,5100) (T(P),P=1,KK)
      WRITE(3,5960)
      WRITE(3,5100)(F(P),P=1,KK)
```

```
      WRITE(3,5995)
      WRITE(3,5100) (T(P),P=1,KK)
      WRITE(3,5980)
      WRITE(3,5100)(Q(P),P=1,KK)
      WRITE(3,5995)
      WRITE(3,5100) (T(P),P=1,KK)
      WRITE(3,5990)
      WRITE(3,5100)( QMEAN(P),P=1,KK)


C                          IF TOL REACHED GO TO 9910 AND PRINT
C                          OTHERWISE CONT  9000 LOOP
      IF(NER4.EQ.0.AND.NN.GE.2) GO TO 9110
  160 LL=NN+1
 9000 CONTINUE
C                          IF NNN TERMS DOES NOT PRODUCE REQUIRED
C                          TOL. SET ERR CODE
 9110 CONTINUE
      WRITE (3,5000)
      IF(NER1.EQ.0) GO TO 9120
      WRITE (3,5900)
      WRITE (3,5910) PLACE1, NER1,REM1
 9120 CONTINUE
      IF (NER2.EQ.0) GO TO 9130
      WRITE (3,5900)
      WRITE (3,5910)PLACE2, NER2, REM2
 9130 CONTINUE
      IF (NER3.EQ.0) GO TO 9140
      WRITE (3,5900)
      WRITE (3,5910) PLACE3, NER3, REM3
 9140 CONTINUE
      IF (NER4.EQ.0) GO TO 9200
      WRITE (3,5900)
      WRITE (3,5910) PLACE4, NER4, REM4
      IF (NG.NE.12) GO TO 10
```

```
 9200 CONTINUE
      IF(NG.NE.12)GO TO 9290
      DO 9280 I=1,NNN
      STORE(I)=GINTP(I)
      STORE1=F(1)
      STORSS=STEADY
 9280 CONTINUE
      GO TO 10
 9290 CONTINUE
      GO TO (9610,9620,9630,9640,9650,9660,9670,9680,9690,9700,9710),NG
 9610 NL = 1
      NU =  5
      GO TO 9800
 9620 NL =  6
      NU = 10
      GO TO 9800
 9630 NL = 11
      NU = 15
      GO TO 9800
 9640 NL = 16
      NU =  20
      GO TO 9800
 9650 NL =  21
      NU =  25
      GO TO 9800
 9660 NL =  26
      NU =  30
      GO TO 9800
 9670 NL =  31
      NU =  35
      GO TO 9800
 9680 NL =  36
      NU =  40
      GO TO 9800
 9690 NL =  41
```

```
      NU =  45
      GO TO 9800
 9700 NL =  46
      NU =  50
      GO TO 9800
 9710 NL =  51
      NU =  55
 9800 CONTINUE
      WRITE (3,5000)
      WRITE (3,5002)
      WRITE (3,5040)(TITLES(I),I=NL,NU)
      WRITE (3,5006)
      I = 1
      LINE = 12
 9850 CONTINUE
      WRITE (3,5200) T(I),QMFAN(I)
      KKK=KK
      IF(I.EQ.KK.OR.I.EQ.KKK)GO TO 9990
      I = I + 1
      LINE = LINE + 1
      IF(LINE.GT.60) GO TO 9900
      GO TO 9850
 9900 CONTINUE
      WRITE (3,5000)
      LINE = 4
      GO TO 9850
 9990 CONTINUE
      STEADY=STEADY+STORSS
      WRITE (3,5005) STEADY
      GO TO 10
      END
```

SUB-ROUTINE SET-UP

```
      SUBROUTINE SETUP
      DIMENSION AT1R(100),AT1(100),AT0(100),ATL(300)
      REAL MU
      COMMON /PARAM/      B(10),BB(10),FF,A,A1,A2,A3,C,D,ZT,C6,C7,TOS,
     1        A23,RK,Z,BE,PRS,CONDK,GAMA,Q,BULK,MU,RO,H,MG,R9,RI,A3ZERO
     2,NB
      COMMON /PART/ATJ(300),NPU,DEL,IMAX
     3       /MAIN / NG,NNN,KK,RRMOVF,SUMDEL
     4       /MAINDB/ T, L1, ACCPT, FOCUS
     3       /BJCOM/ BJDEL
      DOUBLE PRECISION T(101),L1,   ACCPT,FOCUS
4005  FORMAT(4I2)
4010  FORMAT(I2,8X,7D10.5/(8D10.5))
4020  FORMAT(F10.5,F16.10,2I2,F10.5)
4030  FORMAT (5F10.5)

4040  FORMAT (2F10.5,6E10.5)
4050  FORMAT( E10.4,I2)
4060  FORMAT (8E10.4 )
4070  FORMAT(3D10.3,I2)
5000  FORMAT(1H3,7H 012992,20X,23HFLEXIBLE PAVEMENT MODEL)
5002  FORMAT(1H0,17X,45HVISCOELASTIC PLATE ON PORO ELASTIC HALF SPACE)
5004  FORMAT(1H0,19X,19H** INPUT SUMMARY **)
5013  FORMAT(//)
5022  FORMAT(1H ,10X,A6,1H=,F10.5)
5023  FORMAT(1H ,10X,A5,1H=,F16.10)
5024  FORMAT(1H ,10X,A6,1H=,E10.4)
5026  FORMAT(1H ,2X,(7(1X,E10.4)/))
5028  FORMAT(1H ,2X,(7(1X,D10.4)/))
5015  FORMAT ( / )
5020  FORMAT (1H ,10X,A6,1H=,I3)
5030  FORMAT(1H0,38HNO. OF PARTION POINT ZEROES AVAILABLE ,I3)
6000  FORMAT(1H0,23HRR MUST = 0 FOR MG = 11)
      KK=20
      T(1) = 0.D0
      T(2)=1.D-2
      T(3)= 5.D-2
      T(4)=1.D-1
      T(5)=2.5D-1
      T(6)=5.0D-1
      T(7)=1.0D0
      T(8)=1.5D0
      T(9)=2.0D0
      T(10)=2.5D0
      T(11)=3.0D0
      T(12)=3.5D0
      T(13)=4.0D0
      T(14)=4.5D0
      T(15)=5.0D0
      T(16)=7.0D0
      T(17)=1.0D1
      T(18)=1.5D1
      T(19)=2.0D1
      T(20)=3.0D1
      DEL =  .001
      BJDEL=.001
      SUMDEL = .05
      IMAX=8
      NJO=60
      NNN=14
      L1   = 1.D2
      FOCUS = .5D0
      ACCPT = .1D-1
      READ(1,4005,END=900) MG,OPT1,OPT2,OPT3
      IF(OPT1.EQ.1) READ(1,4010) KK,(T(I),I=2,KK)
      IF(OPT2.EQ.1) READ(1,4020) DEL,BJDEL,IMAX,NJO,SUMDEL
      IF(OPT3.EQ.1) READ(1,4070) L1,  ACCPT,FOCUS,NNN
      READ (1,4030) Z,RR,H,BE,Q
      READ (1,4040) FF,R,BULKO,BULK,GAMA,DARCY,A2,A1
      READ (1,4050) RO,   NB
```

```
      READ (1,4060)(B(I),I=1,NB)
      READ (1,4060)(BB(I),I=1,NB)
C                          WRITE INPUT SUMMARY
      NPG = 1
      WRITE(3,5000)
      WRITE(3,5002)
      WRITE(3,5004)
      WRITE(3,5013)
      ID = 6H MG
      WRITE (3,5020) ID,MG
      ID = 6HNNN
      WRITE (3,5020) ID,NNN
      ID = 6H KK
      WRITE (3,5020) ID,KK
      ID = 6H T(I)
      WRITE (3,5020) ID
      WRITE (3,5028)(T(I),I=1,KK)
      WRITE (3,5015)
      ID = 6H DEL
      WRITE (3,5022) ID,DEL
      ID = 6H BJDEL
      WRITE (3,5023) ID,BJDEL
      ID = 6HSUMDEL
      WRITE(3,5022) ID, SUMDEL
      ID = 6H IMAX
      WRITE (3,5020)ID,IMAX
      ID = 6H NJD
      WRITE (3,5020)ID,NJD
      ID = 6H L1
      WRITE (3,5020) ID,L1
      ID = 6H ACCPT
      WRITE (3,5022) ID,ACCPT
      ID = 6H FOCUS
      WRITE (3,5022) ID,FOCUS
      WRITE (3,5015)
```

```
      ID = 6H Z
      WRITE (3,5022) ID,Z
      ID = 6H RR
      WRITE (3,5022) ID,RR
      ID = 6H H
      WRITE (3,5022) ID,H
      ID = 6H BE
      WRITE (3,5022) ID,BE
      ID = 6H Q
      WRITE (3,5022) ID,Q
      WRITE (3,5015)
      ID = 6H FF
      WRITE (3,5022) ID,FF
      ID = 6H R
      WRITE (3,5022) ID,R
      ID = 6H BULKO
      WRITE(3,5024) ID,BULKO
      ID = 6H BULK
      WRITE(3,5024) ID,BULK
      ID = 6H GAMA
      WRITE(3,5024) ID,GAMA
      ID = 6H DARCY
      WRITE(3,5024)ID,DARCY
      ID = 6H A1
      WRITE(3,5024)ID,A1
      ID = 6H A2
      WRITE(3,5024)ID,A2
      WRITE(3,5015)
      ID = 6H RO
      WRITE(3,5024) ID,RO
      ID = 6H B(I)
      WRITE(3,5020) ID
      WRITE(3,5026)(B(I),I=1,NB)
      ID = 6H BB(I)
      WRITE(3,5020) ID
      WRITE(3,5026)(BB(I),I=1,NB)
      IF(RR.NE.0..AND.MG.EQ.11) GO TO 9900
```

```
      IF((RR.EQ.0.).AND.(MG.EQ.8.OR.MG.EQ.9)) NNN=0
      RRMOVE=RR
    2 CONTINUE
      NG = MG
      PI = 3.141593
      A =     GAMA /DARCY
      A = A / FF
      MU = .5
      A3 = BULK - 2.*A2/3.
      A3ZERO = BULKO - 2.* A2/3.
      A23 = 2.*A2+A3
      D = H**3 / ((1.-MU**2)*12.)
      C =  A23        * FF/(A*(A1+1.) * R)
      NO = 0
      N1 = 0
      N1R= 0
C                  ZEROS FOR BJ1(AT*BE)
      AT1(1) = 3.83171 / BE
      N1 = 1
      DO 150 J = 2,NJO
      GAM = 4.*J +1.
      ATB1 = (PI*GAM ) /(4.*BE)
      ATB2 =  1. - 6./(PI**2 * GAM**2) + 6./(PI**4 * GAM**4)
     1         -4761./(5.*PI**6 * GAM**6) + 3902418./(35.*PI**8 * GAM**8)
     2         -8952167324./(35.*PI**10 * GAM**10)
      AT1(J) = ATB1 + ATB2
      N1 = N1 + 1
  150 CONTINUE
      IF(RR.EQ.0.  ) GO TO 800
      IF(MG.EQ.8.OR.MG.EQ.9.OR.MG.EQ.12) GO TO 190
C                  ZEROS FOR BJO(AT*RR)
      ATO(1) = 2.40482 / RR
      NO = 1
      DO 180 J = 2,NJO
      BETA = 4.*J - 1.
      ATR1 =(PI*BETA) /(4.*RR)
      ATR2 =  1. + 2./(PI**2 * BETA**2)  - 62./(3.*PI**4 * BETA**4)
     1      + 15116./(15.*PI**6 * BETA**6) - 12554474./(105.*PI**8 *
     2      BETA**8) + 8368654292./(315.*PI**10 * BETA**10)
      ATO(J) = ATR1 + ATR2
      NO = NO + 1
  180 CONTINUE
      IF (MG.LT.8.OR.MG.EQ.10) GO TO 215
  190 CONTINUE
C                  ZEROS FOR BJ1R(AT*RR)
      AT1R(1) = 3.83171 / RR
      N1R= 1
      DO 215 J = 2,NJO
      GAM = 4.*J +1.
      ATB1 = (PI*GAM ) /(4.*RR)
      ATB2 =  1. - 6./(PI**2 * GAM**2) + 6./(PI**4 * GAM**4)
     1         -4761./(5.*PI**6 * GAM**6) + 3902418./(35.*PI**8 * GAM**8)
     2         -8952167324./(35.*PI**10 * GAM**10)
      AT1R(J) = ATB1 + ATB2
      N1R = N1R + 1
  215 CONTINUE
  400  CONTINUE
      FINAL = AT1(NJO)
      IF (MG.EQ.9.OR.MG.EQ.8.OR.MG.EQ.12) GO TO 700
C ** ARRANGE ATO AND AT1 ARRAYS IN NUMERICAL ORDER AND STORE IN ATJ
      I = 1
      J = 1
      K = 1
      IF(FINAL.GT.ATO(NJO))FINAL=ATO(NJO)
      GO TO 412
  410 CONTINUE
      IF(ATJ(K).EQ.FINAL) GO TO 500
      K = K + 1
  412 CONTINUE
      IF(I.GT.N1) GO TO 420
      IF(J.GT.NO) GO TO 430
```

```
      IF ( ATO(J)-AT1(I) ) 420,430,430
  420 ATJ(K) = ATO(J)
      J = J+1
      GO TO 410
  430 ATJ(K) = AT1(I)
      I = I+1
      GO TO 410
  450 CONTINUE
      ATJ(K) = AT1(I)
      J = J + 1
      I = I+1
      GO TO 410
  500 CONTINUE
      IF (MG.LT. 8.OR.MG.EQ.10) GO TO 1000
C                    STORE ATJ IN ATL, THEN ORDER ATL WITH AT1R
C                    TO GIVE THE FINAL ATJ ARRAY
      DO 505 I=1,K
      ATL(I) = ATJ(I)
  505 CONTINUE
      I = 1
      J = 1
      K = 1
      IF(FINAL.GT.AT1R(NJO))FINAL=AT1R(NJO)
      GO TO 512
  510 CONTINUE
      IF(ATJ(K).EQ.FINAL) GO TO 1000
      K = K+1
  512 CONTINUE
      IF (I.GT.N1R)  GO TO 520
      IF ( ATL(J)-AT1R(I) ) 520,550,530
  520 CONTINUE
      ATJ(K) = ATL(J)
      J = J+1
      GO TO 510
  530 CONTINUE
      ATJ(K) = AT1R(I)
      I = I+1
      GO TO 510
  550 CONTINUE
      ATJ(K) = AT1R(I)
      J = J + 1
      I = I+1
      GO TO 510
C                    ORDER AT1 AND AT1R - STOR IN ATJ
C                    FOR  MG = 8  OR  MG = 9
  700 CONTINUE
      I = 1
      J = 1
      K = 1
      IF(FINAL.GT.AT1R(NJO))FINAL=AT1R(NJO)
      GO TO 712
  710 CONTINUE
      IF(ATJ(K).EQ.FINAL) GO TO 1000
      K = K+1
  712 CONTINUE
      IF(I.GT.N1)   GO TO 720
      IF(J.GT.N1R)  GO TO 730
      IF ( AT1R(J)-AT1(I) ) 720,750,730
  720 CONTINUE
      ATJ(K) = AT1R(J)
      J = J+1
      GO TO 710
  730 CONTINUE
      ATJ(K) = AT1(I)
      I = I+1
      GO TO 710
  750 CONTINUE
      ATJ(K) = AT1(I)
      J = J + 1
      I = I+1
```

```
      GO TO 710
  800 CONTINUE
C                               FOR CASE RR=0
      DO 810 K=1,N1
      ATJ(K) = AT1(K)
  810 CONTINUE
      K = N1
 1000 CONTINUE
      NPC = N1+N0+N1R
      NPU = K
 9000 CONTINUE
      WRITE(3,5030)      NPU
      RETURN
 9900 CONTINUE
      WRITE(3,6000)
  900 CALL TIMER
      END
```

FUNCTION HANKEL

```
      FUNCTION HANKEL(  XS)
      COMMON /PART/ATJ(300),NPU,DEL,IMAX
     &        /ERR/NER1,NER2,NER3,NER4
      DOUBLE PRECISION XS
C ** BEGIN CALCULATION OF SOLUTION BY CALLING SMPSN
      SS = XS
  500 HANKEL = 0.
      ATA=.00001
      DO 550 J=1,NPU
      ATB = ATJ(J)
  520 CALL SMPSN (ATA,ATB,DEL,IMAX,ST1,SOL,NI ,IER,SS)
      HANKEL = HANKEL + SOL
C                     REMOVE "GO TO 530" TO OBTAIN INVERSION
C                             INTEGRAL RESULTS
      GO TO 530
      WRITE(3,9100) NI,IER
      WRITE(3,9000)ATA,ATB
      WRITE(3,9000)HANKEL,SOL
  530 ATA = ATB
      IF (HANKEL.EQ.0.) GO TO 550
      CHECK = SOL / HANKEL
      CHECK = ABS(CHECK)
      IF (CHECK.LT.DEL) GO TO 551
  550 CONTINUE
      NER1=NER1+1
  551 HANKEL = HANKEL - SOL/2.
  570 RETURN
  905 CONTINUE
      WRITE (3,9260)
 9000 FORMAT(1H0,2F14.7)
 9100 FORMAT(1H0,2I4)
 9260 FORMAT (1H0,16HDATA OUT OF SORT)
      STOP
      END
```

SUBROUTINE SMPSN

```
C         SUBROUTINE SMPSN
C
C         PURPOSE
C            INTEGRATES THE GIVEN FUNCTION OVER THE PRESCRIBED RANGE
C
C         USAGE
C            CALL SMPSN(F,A,B,DEL,IMAX,SI1,S,N,IER)
C
C         DESCRIPTION OF PARAMETERS
C            F    -NAME OF USER FUNCTION SUBPROGRAM GIVING F(X)
C            A    -LOWER INTEGRATION LIMIT
C            B    -UPPER INTEGRATION LIMIT
C            DEL  -REQUIRED ACCURACY OR TOLERANCE
C            IMAX-MAXIMUM NUMBER OF RECOMPUTATIONS OF INTEGRAL VALUE
C            SI1  -RESULTANT VALUE OF INTEGRAL JUST PRIOR TO FINAL
C                  EVALUATION
C            S    -RESULTANT FINAL VALUE OF INTEGRAL
C            N    -RESULTANT NUMBER OF INTERVALS USED IN COMPUTING S
C            IER-RESULTANT ERROR CODE WHERE
C                  IER=0   NO ERROR
C                  IER=1   A=B
C                  IER=2   DEL=0
C                  IER=3   IMAX LESS THAN 2
C                  IER=4   REQUIRED ACCURACY NOT MET IN IMAX STEPS
C
C         REMARKS
C            SEE ERROR CODES ABOVE
C
C         SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED
C            F- FUNCTION SUBPROGRAM WHICH COMPUTES F(X) FOR X BETWEEN
C            A AND B.
C            CALLING PROGRAM MUST HAVE FORTRAN EXTERNAL STATEMENT
C            CONTAINING NAMES OF FUNCTION SUBPROGRAMS LISTED IN CALL TO
C            SMPSN
C
```

```
C         METHOD
C            SIMPSON*S RULE IS PERFORMED WITH INTERVAL HALVING UNTIL
C            DIFFERENCE BETWEEN SUCCESSIVE VALUES OF THE INTEGRAL IS
C            LESS THAN DEL.  FAILURE TO REACH THE TOLERANCE AFTER IMAX
C            TRIES TERMINATES THE SUBROUTINE EXECUTION.
      SUBROUTINE SMPSN(  A,B,DEL,IMAX,SI1,S,NI,IER,SS)
C
C
C        IF A DOUBLE PRECISION VERSION OF THIS ROUTINE IS DESIRED, THE
C        C IN COLUMN 1 SHOULD BE REMOVED FROM THE DOUBLE PRECISION
C        STATEMENT WHICH FOLLOWS.
C
C     DOUBLE PRECISION A,B,DEL,SI1,S,BA,X,SUMK,FRSTX,XK,FINC,F
C
C        THE C MUST ALSO BE REMOVED FROM DOUBLE PRECISION STATEMENTS
C        APPEARING IN OTHER ROUTINES USED IN CONJUNCTION WITH THIS
C        ROUTINE.
C
C        THE DOUBLE PRECISION VERSION OF THIS SUBROUTINE MUST ALSO
C        CONTAIN DOUBLE PRECISION FORTRAN FUNCTIONS.  ABS IN STATEMENT
C        27 MUST BE CHANGED TO DABS.
C
C        USER FUNCTION SUBPROGRAM, F, MUST BE IN DOUBLE PRECISION.
C
C
      COMMON /ERR/NER1,NER2,NER3,NER4
      SI1=.0
      S=.0
      N=0
      BA=B-A
      IF (BA)20,19,20
   19 IER=1
      RETURN
   20 IF(DEL)22,22,23
```

```
   22 IER=2
      RETURN
   23 IF(IMAX-1)24,24,25
   24 IER=3
      RETURN
C
C     COMPUTE SIGMA(1)
C
   25 X=BA/2.+A
      NHALF=1
      SUMK = G(X,SS) *BA * 2./ 3.
      S =      SUMK + ( G(A,SS) + G(B,SS)) * BA / 6.
C
C     DIVIDE (A,B) INTO 2,4,...,2**I INTERVALS.
C     COMPUTE SIGMA(2),SIGMA(4),...,SIGMA(I)
C
      DO 28 I=2,IMAX
      SI1=S
      S=(S-SUMK/2.)/2.
      NHALF=NHALF*2
      ANHLF=NHALF
      FRSTX=A+(BA/ANHLF)/2.
      SUMK=G(FRSTX,SS)
      XK=FRSTX
      KLAST=NHALF-1
      FINC=BA/ANHLF
      DO 26 K=1,KLAST
      XK=XK+FINC
   26 SUMK=SUMK+G(XK,SS)
      SUMK=SUMK*2.*BA/(3.*ANHLF)
      S=S+SUMK
C
C     COMPARES THE I-TH AND(I-1)ST RESULTS
C
   27 IF( ABS(S-SI1)- ABS(DEL*S)) 29,29,28
   28 CONTINUE
      NER2 = NER2 + 1.
      IER=4
      GO TO 30
   29 IER=0
   30 NI=2*NHALF
      RETURN
      END
```

FUNCTIONS G THROUGH G12

```
      FUNCTION G(AT,S)
      COMMON /PARAM/       B(10),BB(10),FF,A,A1,A2,A3,C,D,ZT,C6,C7,TOS,
     1        A23,RK,Z,BE,PRS,CONDK,GAMA,Q,BULK,WL,RO,H,WG,RB,RI,A3ZERO
     2,NB
     2        /MODE/ GCODE
     3        /GFUN/ RKSS,BJORR,BJ1RR,BJ1BE,ZTAT,EXPATZ,EXPZTZ,C5,
     4              G2ATS,G4ATS,G5ATS,G6ATS,G7ATS,G10ATS
      IF (S.GT.0.0    ) GO TO 10
  502 FORMAT (1H ,3HAT=,E16.10,3H S=,E16.10)
      WRITE(3,502) AT,S
      STOP
   10 CONTINUE
      XBJ = RB*AT
      BJORR = BJ(XBJ,0)
      BJ1RR = BJ(XBJ,1)
      XBJ = BE*AT
      BJ1BE = BJ(XBJ,1)
      AT2SC = AT**2 + S/C
      ZT = SQRT(AT2SC)
      ZTAT = S / C
      ATZ = -AT * Z
      ZTZ = -ZT * Z
      EXPATZ = EXP(ATZ)
      EXPZTZ = EXP(ZTZ)
      IF (AT.GT.0) GO TO 12
      RKSS = (A2*Q*BF**2)/(2.*A2*(A3+A2)*C)
      RK = (A2 * Q * BF **2) / (2.*A2 * (A3ZERO+A2)*C )
      GO TO 14
   12 CONTINUE
      RK=2.*A2*BE            *Q * BJ1BE       /
     1   (C*D*RO*AT**4 * (A3ZERO+2.*A2)+2.*A2*C*(A3ZERO +A2)*AT)
   14 CONTINUE
      TOSO = 0.
      TOS = 0.
      DO 20 I=1,NB
      TOSO = TOSO + BB(I)
      TOS =  TOS + BB(I)        /     (S+B(I))
   20 CONTINUE
      RI = RO - TOSO
      C6 =RK * A2 * (2.*AT+ZT) /(ZT * (ZT+AT)**2)
      C7 = ((2.*A2)**2 * AT**3 - 2.*A2*AT**2 * ZT*A23-2.*A2*AT*ZT**2*A23
     1   )/ (ZT * ( ZT + AT ) * A23)
   50 CONTINUE
      IF(GCODE.EQ.3) GO TO 90
      IF(AT.EQ.0.) GO TO 51
      D3 = (C6-G1(AT,S)) / C7
      C3 = RK / (2.* AT * (     ZTAT      ))
      C5 =(G1(AT,S) - 2.*A2*AT*(D3+C3)) /(2.*A2*AT*(-ZTAT))*(ZT+AT)
      RKSS =  (2.*A2 * Q * BE * BJ1BE)  /
     1          (C * ( 2.* A2 * AT *(A2+A3) + D * RI * AT**4*(2.*A2+A3)))
   51 CONTINUE
   90 CONTINUE
      GO TO (100,200,300,400,500,600,700,800,900,1000,1100,1200),WG
  100 G = G1(AT,S)
      RETURN
  200 G = G2(AT,S)
      RETURN
  300 CONTINUE
      IF (GCODE.EQ.3.) GO TO  350
      G2ATS=  G2(AT,S)
  350 CONTINUE
      G = G3(AT,S)
      RETURN
  400 CONTINUE
      IF (GCODE.EQ.3.) GO TO  450
      G2ATS = G2(AT,S)
      G5ATS = G5(AT,S)
  450 CONTINUE
      G = G4(AT,S)
      RETURN
```

```
  500 CONTINUE
      G = G5(AT,S)
      RETURN
  600 CONTINUE
      IF (GCODE.EQ.3.) GO TO  650
      G2ATS = G2(AT,S)
      G5ATS = G5(AT,S)
  650 CONTINUE
      G = G6(AT,S)
      RETURN
  700 CONTINUE
      IF (GCODE.EQ.3.) GO TO  750
      G2ATS = G2(AT,S)
      G5ATS = G5(AT,S)
      G6ATS = G6(AT,S)
  750 CONTINUE
      G = G7(AT,S)
      RETURN
  800 CONTINUE
      IF (GCODE.EQ.3.) GO TO  850
      G2ATS = G2(AT,S)
  850 CONTINUE
      G = G8(AT,S)
      RETURN
  900 CONTINUE
      IF (GCODE.EQ.3.) GO TO  950
      G2ATS = G2(AT,S)
  950 CONTINUE
      G = G9(AT,S)
      RETURN
 1000 CONTINUE
      IF (GCODE.EQ.3.) GO TO 1050
      G2ATS = G2(AT,S)
      G5ATS = G5(AT,S)
      G6ATS = G6(AT,S)
      G7ATS = G7(AT,S)
      G4ATS = G4(AT,S)
 1050 CONTINUE
      G = G10(AT,S)
      RETURN
 1100 CONTINUE
      IF (GCODE.EQ.3.) GO TO 1150
      G2ATS = G2(AT,S)
      G5ATS = G5(AT,S)
      G6ATS = G6(AT,S)
      G7ATS = G7(AT,S)
      G4ATS = G4(AT,S)
      G10ATS=G10(AT,S)
 1150 CONTINUE
      G = G11(AT,S)
      RETURN
 1200 CONTINUE
      IF (GCODE.EQ.3.) GO TO 1250
      G2ATS = G2(AT,S)
      G5ATS = G5(AT,S)
      G6ATS = G6(AT,S)
      G7ATS = G7(AT,S)
      G4ATS = G4(AT,S)
 1250 CONTINUE
      G=G12(AT,S)
      RETURN
      END


      FUNCTION G1(AT,S)
      COMMON /PARAM/      R(10),RB(10),FF,A,A1,A2,A3,C,D,ZT,C6,C7,TCS,
     1         A23,RK,Z,BE,PRS,CONDK,GAMA,Q,BULK,WL,RO,H,VG,RR,RI,A37ERO
     2,NR
     2        /MODE/ GCODE
     3        /GFUN/ RKSS,RJORR,BJ1RR,RJ1BE,ZTAT,EXPATZ,EXPZTZ,C5,
```

```
     4              G2ATS,G4ATS,G5ATS,G6ATS,G7ATS,G10ATS
      IF (AT.GT.0) GO TO 50
      GO TO (10,20,30),GCODE
   10 CONTINUE
   20 CONTINUE
   30 CONTINUE
C                          TIME = 0 , AT = 0
      G1=0.
      RETURN
   50 CONTINUE
      CORX = -RKSS * C * (A2+A3) / S
      GO TO (100,200,300),GCODE
  100 CONTINUE
C
      G1A= D * (RI + S*TOS)*AT**4
      G1C = C7 * Q * BE* BJ1BE    /(AT * S*( C7 -          G1A ))
      G1B = -C6           * G1A /((C7 -          G1A)
      G1 = G1B - G1C
C
C                        ABOVE IS VALUE OF G1 FOR USE IN OTHER G-FUNCTI
      IF (MG.NE.1) RETURN
C                        BELOW IS TIME DEPENDENT CASE
C
      G1 = G1 * AT * BJORR
      RETURN
  200 CONTINUE
C
C                          TIME INDEPENDENT CASE
C
      G1 = CORX*AT*BJORR
      RETURN
  300 CONTINUE
C                          G1 FOR TIME = 0
      G1 = (-RK * C * (A3ZERO+A2)) * BJORR * AT / S
       RETURN
      END
```

```
      FUNCTION G2(AT,S)
      COMMON /PARAM/       B(10),BB(10),FF,A,A1,A2,A3,C,D,ZT,C6,C7,TOS,
     1        A23,RK,Z,BE,PRS,CONDK,GAMA,Q,BULK,MU,RO,H,WG,RR,RI,A3ZERO
     2,NB
     2        /MODE/ GCODE
     3        /GFUN/ RKSS,BJORR,BJ1RR,BJ1BE,ZTAT,EXPATZ,EXPZT7,C5,
     4              G2ATS,G4ATS,G5ATS,G6ATS,G7ATS,G10ATS
      IF (AT.GT.0) GO TO 50
      GO TO (10,20,30),GCODE
   10 CONTINUE
C
C                          TIME DEPENDENT, AT=0  CASE
C
      G2 =  Q * BE*BF * (2.*A2+A3ZERO) / (4.*A2*S*(A2+A3ZERO))
      CORX=Q*BE*BE*(2.*A2+A3)/(4.*A2*S*(A2+A3))
      RETURN
   20 CONTINUE
C
C                          TIME INDEPENDENT, AT=0 CASE
C
      G2 =  Q * BE*BE * (2.*A2+A3)      / (4.*A2*S*(A2+A3))
      RETURN
   30 CONTINUE
C                          TIME = 0 , AT = 0
      G2 =  Q * BE*BF * (2.*A2+A3ZERO) / (4.*A2*S*(A2+A3ZERO))
      RETURN
   50 CONTINUE
      CORX =  RKSS * C * (2.*A2+A3) / (2.* A2 * AT * S)
      GO TO (100,200,300),GCODE
  100 CONTINUE
C
      G2 = (G1(AT,S)-C6) / C7
  120 FORMAT (1H ,3HG2=,E16.10)
C
C                          ABOVE IS G2 FOR USE IN OTHER G-FUNCTIONS
```

```
      IF (MG.NE.2)  RETURN
C                             TIME DEPENDENT, AT=POSITIVE
C
      G2 =G2*AT*BJORR
      RETURN
  200 CONTINUE
C                             TIME INDEPENDENT, AT=POSITIVE
C
      G2 = CORX * AT*BJORR
      RETURN
  300 CONTINUE
C                             G2  FOR TIME = 0
      G2TO =  RK * C * (2.* A2 + A3ZERO) / (2.* A2 * AT * S)
      G2 = G2TO * AT * BJORR
      RETURN
      END



      FUNCTION G3(AT,S)
      COMMON /PARAM/      B(10),BB(10),FF,A,A1,A2,A3,C,D,ZT,C6,C7,TOS,
     1       A23,RK,Z,BE,PRS,CONDK,GAMA,Q,BULK,MU,RO,H,MG,RR,RI,A3ZERO
     2,NR
     2       /MODE/ GCODE
     3       /GFUN/ RKSS,BJORR,BJ1RR,BJ1RE,ZTAT,EXPATZ,EXPZTZ,C5,
     4             G2ATS,G4ATS,G5ATS,G6ATS,G7ATS,G10ATS
      IF (AT.GT.0) GO TO 50
      GO TO (10,20,30),GCODE
   10 CONTINUE
C
C                             TIME DEPENDENT , AT=0
      G3 = (Q * RE*RE /(4.*S))*(1./ A2 + 1./(A2+A3ZERO))
      CORX=Q*BE*BE*(2.*A2+A3)/(4.*A2*S*(A2+A3))
      RETURN
   20 CONTINUE
C
C                             TIME INDEPENDENT , AT = 0
      G3 =  Q * BE*BE * (2.*A2+A3)     / (4.*A2*S*(A2+A3))
      RETURN
   30 CONTINUE
C                             TIME = 0 , AT = 0
      G3 =  Q * RE*RE * (2.*A2+A3ZERO) / (4.*A2*S*(A2+A3ZERO))
      RETURN
   50 CONTINUE
C
C                             TIME DEPENDENT , AT = POSITIVE
      CORX = (RKSS   *   (2.*A2+A3) * (1.+AT*Z) / (A2+AT) - RK * Z )
     1                            *  EXPATZ * C / (2.* S)
      GO TO (100,200,300),GCODE
  100 CONTINUE
C
      G3A = (ZT*C5 + G2ATS    ) * EXPATZ
      G3B = (ZT*C5)              * EXPZTZ
      G3C = ( RK*C/(2.*S)            - AT*G2ATS    ) * EXPATZ    *Z
C                             G3 FOR USE IN OTHER FUNCTION
      G3 = G3A - G3B - G3C
      IF (MG.NE.3)  RETURN
      G3=G3*AT*BJORR
  600 FORMAT (1H ,4(1X,E16.10))
  190 CONTINUE
      RETURN
  200 CONTINUE
C
C                             TIME INDEPENDENT , AT = POSITIVE
      G3 = CORX * AT * BJORR
      RETURN
C                             G3  FOR TIME = 0
  300 CONTINUE
      G3TO =  RK * C * EXPATZ * ((2.*A2+A3ZERO)*(1.*AT*Z)/(2.*A2*AT)
     1     -Z/2. ) / S
      G3 = G3TO * AT * BJORR
```

```
      RETURN
      END



      FUNCTION G4(AT,S)
      COMMON /PARAM/       B(10),BB(10),FF,A,A1,A2,A3,C,D,ZT,C6,C7,TOS,
     1        A23,RK,Z,RE,PRS,CONDK,GAMA,Q,BULK,MU,RO,H,WG,RR,RI,A3ZERO
     2,NR
     2        /MODE/ GCODE
     3        /GFUN/ RKSS,BJORR,RJ1RR,BJ1RE,ZTAT,EXPATZ,EXPZTZ,C5,
     4               G2ATS,G4ATS,G5ATS,G6ATS,G7ATS,G10ATS
      IF (AT.GT.0) GO TO 50
C                          TIME DEPENDENT , AT=0
C                          TIME INDEPENDENT , AT = 0
C                          TIME = 0 , AT = 0
      G4 = 0.
      RETURN
   50 CONTINUE
      CORX = (-RKSS *(AT*Z *(2.*A2+A3) +A2+A3)*RK + A2 *     AT*Z)
     1           *  C * EXPATZ / S
      GO TO (100,200,300),GCODE
  100 CONTINUE
C                          TIME DEPENDENT , AT = POSITIVE
C
      G4A = (A3-2.*A1*A2)    *    G5ATS     / (1.+A1)
      G4B1 = (-RK     /  (2.*ZTAT)) -  (ZT + AT   + C5)
     1       -  A1 + AT  + G2ATS     /    (1.+A1)
      G4B1 = G4B1 * EXPATZ
      G4B2 =    (AT + RK/(ZTAT*2.)-AT*AT*G2ATS) + Z*EXPATZ
      G4B3 = ZT**2    *   C5 * EXPZTZ
      G4B = 2.*A2* (G4B1 + G4B2 + G4B3)
C                          G4 FOR USE IN OTHER FUNCTION
      G4 = G4A + G4B
  703 FORMAT(1H ,3HG4=,E16.10)
      IF (MG.NE.4)  RETURN
      G4=G4*AT*BJORR
      RETURN
  200 CONTINUE
C
C                          TIME INDEPENDENT , AT = POSITIVE
      G4 = CORX * AT * BJORR
      RETURN
  300 CONTINUE
C                          G4  FOR TIME = 0
      G4T0 =  -RK * C * EXPATZ + (AT*Z*(A2+A3ZERO)+(A3+A1*A3ZERO)/
     1          (1.+A1) + A2) / S
      G4 = G4T0 * AT * BJORR
      RETURN
      END



      FUNCTION G5(AT,S)
      COMMON /PARAM/       B(10),BB(10),FF,A,A1,A2,A3,C,D,ZT,C6,C7,TOS,
     1        A23,RK,Z,RE,PRS,CONDK,GAMA,Q,BULK,MU,RO,H,WG,RR,RI,A3ZERO
     2,NR
     2        /MODE/ GCODE
     3        /GFUN/ RKSS,BJORR,RJ1RR,BJ1RE,ZTAT,EXPATZ,EXPZTZ,C5,
     4               G2ATS,G4ATS,G5ATS,G6ATS,G7ATS,G10ATS
      IF (AT.GT.0) GO TO 50
C                          TIME DEPENDENT , AT=0
C                          TIME INDEPENDENT , AT = 0
C                          TIME = 0 , AT = 0
      G5 = 0.
      RETURN
   50 CONTINUE
      CORX = -RKSS * C * EXPATZ / S
      GO TO (100,200,300),GCODE
  100 CONTINUE
C
C                          TIME DEPENDENT , AT = POSITIVE
```

```
      G5  =                         ZTAT*C5   * EXPZTZ
     1        * RK * EXPATZ / ZTAT
C                           G5 FOR USE IN OTHER FUNCTION
  703 FORMAT(1H ,3HG5=,E16.10,E16.10)
      IF (MG.NE.5)  RETURN
      G5=G5*AT*BJORR
      RETURN
  200 CONTINUE
C                           TIME INDEPENDENT , AT = POSITIVE
      G5 = CORX * AT * BJORR
      RETURN
      GO TO (100,200,300),GCODE
  300 CONTINUE
C                           G5  FOR TIME = 0
      G5TO = -RK * C * EXPATZ / S
      G5 = G5TO * AT * BJORR
      RETURN
      END



      FUNCTION G6(AT,S)
      COMMON /PARAM/       B(10),BB(10),FF,A,A1,A2,A3,C,D,ZT,C6,C7,TOS,
     1      A23,RK,Z,BE,PRS,CONDK,GAMA,Q,BULK,WU,RO,H,WG,RR,RI,A3ZERO
     2,NR
     2       /MODE/ GCODE
     3       /GFUN/ RKSS,BJORR,BJ1RR,BJ1RE,ZTAT,EXPATZ,EXPZTZ,C5,
     4             G2ATS,G4ATS,G5ATS,G6ATS,G7ATS,G10ATS
      IF (AT.GT.0) GO TO 50
C                           TIME = 0 , AT = 0
C                           TIME DEPENDENT , AT=0
C                           TIME INDEPENDENT , AT = 0
      G6 = 0.
      RETURN
   50 CONTINUE
      CORX=.0
```

```
      GO TO (100,200,300),GCODE
  100 CONTINUE
C                           TIME DEPENDENT , AT = POSITIVE
      G6B =  (A3+2.*A2) * G5ATS    /(2.*A2)
      G6A =             AT * G2ATS * EXPATZ
C                           G6 FOR USE IN OTHER FUNCTION
      G6 =(G6A + G6B)*2.*A2/(1.+A1)
  703 FORMAT(1H ,3HG6=,E16.10)
      IF (MG.NE.6)  RETURN
      G6=-G6*AT*BJORR
      RETURN
  200 CONTINUE
C                           TIME INDEPENDENT , AT = POSITIVE
      G6 =-CORX
      RETURN
  300 CONTINUE
C                           G6  FOR TIME = 0
      G6TO = RK * C * EXPATZ * (-A3 + A3ZERO) / ((1.+A1)*S)
      G6 =-G6TO * AT * BJORR
      RETURN
      END



      FUNCTION G7(AT,S)
      COMMON /PARAM/       B(10),BB(10),FF,A,A1,A2,A3,C,D,ZT,C6,C7,TOS,
     1      A23,RK,Z,BE,PRS,CONDK,GAMA,Q,BULK,WU,RO,H,WG,RR,RI,A3ZERO
     2,NR
     2       /MODE/ GCODE
     3       /GFUN/ RKSS,BJORR,BJ1RR,BJ1RE,ZTAT,EXPATZ,EXPZTZ,C5,
     4             G2ATS,G4ATS,G5ATS,G6ATS,G7ATS,G10ATS
      IF (AT.GT.0) GO TO 50
C                           TIME DEPENDENT , AT=0
C                           TIME INDEPENDENT , AT = 0
C                           TIME = 0 , AT = 0
      G7 = 0.
```

```
      RETURN
   50 CONTINUE
      CORX = -RKSS * C * (2.*A2+ 3.* A3) * EXPATZ / S
      GO TO (100,200,300),GCODE
  100 CONTINUE
C
C
C                             TIME DEPENDENT , AT = POSITIVE
      G7 = -3.*A1* G6ATS    + (2.*A2 + 3.*A3) * G5ATS
  703 FORMAT(1H ,3HG7=,E16.10)
C                             G7 FOR USE IN OTHER FUNCTION
      IF (MG.NE.7)  RETURN
      G7=G7*AT*BJORR
      RETURN
  200 CONTINUE
C                             TIME INDEPENDENT , AT = POSITIVE
      G7 = CORX * AT * BJORR
      RETURN
  300 CONTINUE
      G7TO = -RK * C * EXPATZ * (A1*(2.* A2 + 3.* A3ZERO) +
     1         2.* A2 + 3.* A3) /((1.+A1)*S)
      G7 = G7TO * AT * BJORR
C                             G7  FOR TIME = 0
      RETURN
      END


      FUNCTION G8(AT,S)
      COMMON /PARAM/        B(10),BB(10),FF,A,A1,A2,A3,C,D,ZT,C6,C7,TOS,
     1        A23,RK,Z,BE,PRS,CONDK,GAMA,Q,BULK,MU,RO,H,MG,RR,RI,A3ZERO
     2,NR
     2        /MODE/ GCODE
     3        /GFUN/ RKSS,BJORR,BJ1RR,BJ1BE,ZTAT,EXPATZ,EXPZTZ,C5,
     4                G2ATS,G4ATS,G5ATS,G6ATS,G7ATS,G10ATS
      IF (AT.GT.0) GO TO 50
```

```
C                             TIME DEPENDENT , AT=0
C                             TIME INDEPENDENT , AT = 0
C                             TIME = 0 , AT = 0
      G8 = 0.
      RETURN
   50 CONTINUE
      CORX = ( RKSS * AT*Z * (2.*A2+A3) - RKSS*A2 - RK*A2 * AT*Z )
     1             * C * EXPATZ / (2.*A2*AT*S)
      GO TO (100,200,300),GCODE
  100 CONTINUE
C
      G8A =  ZT*C5  -  RK /(2.*AT*ZTAT)  - RK * Z /(2.*ZTAT)*AT*Z*G2ATS
      G8A = G8A * EXPATZ
      G8B = -AT * C5  * EXPZTZ
C                             TIME DEPENDENT , AT = POSITIVE
C                             G8 FOR USE IN OTHER FUNCTION
      G8 =  G8A + G8B
  703 FORMAT(1H ,3HG8=,E16.10)
      IF (MG.NE.8)  RETURN
      G8=G8*AT*BJ1RR
      RETURN
  200 CONTINUE
C                             TIME INDEPENDENT , AT = POSITIVE
      G8 = CORX * AT * BJ1RR
      RETURN
  300 CONTINUE
C                             G8  FOR TIME = 0
      G8TO = RK * EXPATZ * C * ((A2+A3ZERO)*AT*Z - A2) /(2.* A2 * AT *S)
      G8 = G8TO * AT * BJ1RR
      RETURN
      END


      FUNCTION G9(AT,S)
      COMMON /PARAM/       B(10),BB(10),FF,A,A1,A2,A3,C,D,ZT,C6,C7,TOS,
```

```
1        A23,RK,Z,RF,PRS,CONDK,GAMA,Q,BULK,WL,RO,H,WG,RR,RI,A37ERO
2,NR
2        /MODE/ GCODE
3        /GFUN/ RKSS,BJORR,BJ1RR,BJ1RE,ZTAT,EXPATZ,EXPZTZ,C5,
4                G2ATS,G4ATS,G5ATS,G6ATS,G7ATS,G10ATS
      IF(Z.GT.0..AND.RR.GT.0) GO TO 40
      G9=0.
      RETURN
 40   CONTINUE
      IF (AT.GT.0) GO TO 50
C                       TIME DEPENDENT , AT=0
C                       TIME INDEPENDENT , AT = 0
C                       TIME = 0 , AT = 0
      G9 = 0.
      RETURN
  50  CONTINUE
      CORX = ( RK*A2  -  RKSS   * (2.*A2+A3) ) * C * EXPATZ *Z*AT/S
      GO TO (100,200,300),GCODE
 100  CONTINUE
C
C                       TIME DEPENDENT , AT = POSITIVE
C                       G9 FOR USE IN OTHER FUNCTION
      G9A = AT*G2ATS*Z + ZT*C5 - RK*Z  /(2.*ZTAT)
      G9A = G9A * EXPATZ
      G9B =-ZT * C5 * EXPZTZ
      G9 =  G9A + G9B
      G9 = -2.* A2 * AT * G9
      IF (MG.NE.9)  RETURN
 703  FORMAT(1H ,3HG9=,E16.10)
      G9 = G9 * AT * BJ1RR
      RETURN
 200  CONTINUE
C                       TIME INDEPENDENT , AT = POSITIVE
      G9 = CORX * AT * BJ1RR
      RETURN
```

```
 300  CONTINUE
C                       G9  FOR TIME = 0
      G9TO = -RK * C * AT * Z * EXPATZ * (A2 + A3ZERO) / S
      G9 = G9TO * AT * BJ1RR
      RETURN
      END


      FUNCTION G10(AT,S)
      COMMON /PARAM/      B(10),BB(10),FF,A,A1,A2,A3,C,D,ZT,C6,C7,TOS,
1        A23,RK,Z,RE,PRS,CONDK,GAMA,Q,BULK,WL,RO,H,WG,RR,RI,A37ERO
2,NR
2        /MODE/ GCODE
3        /GFUN/ RKSS,BJORR,BJ1RR,BJ1RE,ZTAT,EXPATZ,EXPZTZ,C5,
4                G2ATS,G4ATS,G5ATS,G6ATS,G7ATS,G10ATS
      IF(AT.GT.0.) GO TO 50
C                       TIME DEPENDENT , AT = 0
C                       TIME INDEPENDENT , AT = 0
C                       TIME = 0 , AT = 0
      G10= 0.
      RETURN
 50   CONTINUE
      IF (RR.GT.0) GO TO 60
      CORX4= (-RKSS *(AT*Z *(2.*A2+A3) +A2+A3)+RK * A2 *    AT*Z)
1             *  C * EXPATZ / S
      CORX7= -RKSS * C * (2.*A2+ 3.* A3) * EXPATZ / S
      CORX=(CORX7-CORX4)*AT/2.
      GO TO 80
 60   CONTINUE
      COR2=-RKSS*((2.*A2+A3)*(1.-AT*Z)-A2)-RK*A2*AT*Z
      CORX=COR2*BJORR
      CORX=CORX*C*EXPATZ/S
      CORX=CORX*AT
 80   CONTINUE
      GO TO (100,200,300),GCODE
```

```
  100 CONTINUE
C                          TIME DEPENDENT , AT = POSITIVE
      IF(RR.GT.0. ) GO TO 150
      G10 = (G7ATS - G4ATS) * AT/2.
       RETURN
  150 CONTINUE
      DG9DZ=((-RK*C + AT*RK*Z*C)/ (2.* S) - ZT*AT*C5 -Z*AT*AT*G2ATS
     1      + AT*G2ATS)  *  EXPATZ   + ZT*ZT*C5 * EXPZTZ
      DG9DZ = DG9DZ *(-2.)*A2*AT
      G10A = (AT + G6ATS + DG9DZ) * RJORR
      G10 = G10A
       RETURN
  200 CONTINUE
C                          TIME INDEPENDENT , AT = POSITIVE
      G10 = CORX
       RETURN
  300 CONTINUE
C                          G10 FOR TIME = 0
      IF(RR.GT.0.) GO TO 350
      G4TO =  -RK * C * EXPATZ * (AT*Z*(A2+A3ZERO)+(A3+A1*A3ZERO)/
     1        (1.+A1) + A2) / S
      G7TO = -RK * C * EXPATZ * (A1*(2.* A2 + 3.* A3ZERO) +
     1        2.* A2 + 3.* A3) /((1.+A1)*S)
      G10=(G7TO-G4TO)*AT/2.
      RETURN
  350 CONTINUE
      G10TOA =  RK * AT * C * EXPATZ / S  * ((-A3+A3ZERO)/(1.+ A1)
     1         - (A2 + A3ZERO) * (1.- AT*Z))  * RJORR
      G10 = G10TOA
      RETURN
      END


      FUNCTION G11(AT,S)
      COMMON /PARAM/      R(10),RR(10),FF,A,A1,A2,A3,C,D,ZT,C6,C7,TCS,
     1      A23,RK,Z,RE,PRS,CONDK,GAMA,Q,BULK,WL,RD,H,WG,RR,RI,A3ZERO
     2,NR
     2      /MODE/ GCODE
     3      /GFUN/ RKSS,BJORR,RJ1RR,BJ1RE,ZTAT,EXPATZ,EXPZTZ,C5,
     4            G2ATS,G4ATS,G5ATS,G6ATS,G7ATS,G10ATS
      IF(AT.GT.0.) GO TO 50
C                          TIME DEPENDENT , AT = 0
C                          TIME INDEPENDENT , AT = 0
C                          TIME = 0 , AT = 0
      G11= 0.
      RETURN
   50 CONTINUE
      CORX4= (-RKSS *(AT*Z *(2.*A2+A3) +A2+A3)+RK * A2 *    AT*Z)
     1         *  C * EXPATZ / S
      CORX7= -RKSS * C * (2.*A2+ 3.* A3) * EXPATZ / S
      CORX=(CORX7-CORX4)*AT/2.
 2000 FORMAT(1H,10E10.4)
      GO TO (100,200,300),GCODE
  100 CONTINUE
C                          TIME DEPENDENT , AT = POSITIVE
      G11 =  (G7ATS-G4ATS) * AT * RJORR  -  G10ATS
      RETURN
  200 CONTINUE
C                          TIME INDEPENDENT , AT = POSITIVE
      G11 = CORX
       RETURN
  300 CONTINUE
C                          G11 FOR TIME = 0
      G4TO =  -RK * C * EXPATZ * (AT*Z*(A2+A3ZERO)+(A3+A1*A3ZERO)/
     1        (1.+A1) + A2) / S
      G7TO = -RK * C * EXPATZ * (A1*(2.* A2 + 3.* A3ZERO) +
     1        2.* A2 + 3.* A3) /((1.+A1)*S)
      G11=(G7TO-G4TO)*AT/2.
      RETURN
      RETURN
      END
```

```
      FUNCTION G12(AT,S)
      COMMON /PARAM/,      R(10),RR(10),FF,A,A1,A2,A3,C,D,ZT,C6,CT,TCS,
     1         A23,RK,Z,RE,PRS,CONDK,GAMA,Q,BULK,WL,RO,H,WG,RR,RI,A37FRO
     2,NR
     2        /MODE/ GCODE
     3        /GFUN/ RKSS,BJORR,RJ1RR,BJ1RE,ZTAT,EXPATZ,EXPZTZ,C5,
     4               G2ATS,G4ATS,G5ATS,G6ATS,G7ATS,G10ATS
      IF(AT.GT.0.) GO TO 50
C                     TIME DEPENDENT , AT = C
C                     TIME INDEPENDENT , AT = 0
C                     TIME = 0 , AT = 0
      G12= 0.
       RETURN
   50 CONTINUE
      IF (RR.GT.0) GO TO 60
      CORX4= (-RKSS *(AT*Z *(2.*A2+A3) +A2+A3)+RK * A2 *     AT*Z)
     1          *  C * EXPATZ / S
      CORX7= -RKSS * C * (2.*A2+ 3.* A3) * EXPATZ / S
      CORX=(CORX7-CORX4)*AT/2.
      GO TO 80
   60 CONTINUE
      COR1=-RKSS*((A2+2.*A3)-(2.*A2+A3)*AT*Z)-RK*A2*AT*Z
      COR2=-RKSS*((2.*A2+A3)*(1.-AT*Z)-A2)-RK*A2*AT*Z
      COR2=COR2*AT
      CORX=COR1*RJ1RR/RR-COR2*(2.*BJ1RR/(AT*RR))
      CORX=CORX*C*EXPATZ/S
   80 CONTINUE
      GO TO (100,200,300),GCODE
  100 CONTINUE
C                     TIME DEPENDENT , AT = POSITIVE
      IF(RR.GT.0. ) GO TO 150
      G12 = (G7ATS - G4ATS) * AT/2.
       RETURN
  150 CONTINUE
      DG9DZ=((-RK*C + AT*RK*Z*C)/ (2.* S) - ZT*AT*C5 -Z*AT*AT*G2ATS
     1         + AT*G2ATS)  *  EXPATZ   + ZT*ZT*C5 * EXPZTZ
      DG9DZ = DG9DZ *(-2.)*A2*AT
      G10A = (AT * G6ATS + DG9DZ)
      G10B = (G7ATS - 2.* G6ATS  - G4ATS - 2.* DG9DZ / AT ) * RJ1RR/RR
      G12 = G10B
       RETURN
  200 CONTINUE
C                     TIME INDEPENDENT , AT = POSITIVE
      G12 = CORX
       RETURN
  300 CONTINUE
C                     G10 FOR TIME = 0
      IF(RR.GT.0.) GO TO 350
      G4TO =  -RK * C * EXPATZ * (AT*Z*(A2+A3ZERO)+(A3+A1*A3ZERO)/
     1          (1.+A1) + A2) / S
      G7TO = -RK * C * EXPATZ * (A1*(2.* A2 + 3.* A3ZERO) +
     1         2.* A2 + 3.* A3) /((1.+A1)*S)
      G12=(G7TO-G4TO)*AT/2.
      RETURN
  350 CONTINUE
      G10A =(A2 + A37ERO)*(AT*Z   ) - A2
      G10TOB = G10A*          RK * C  * EXPATZ * RJ1RR /(-RR * S)
      G12 = G10TOB
      RETURN
      END
```

FUNCTION BJ

```
      FUNCTION BJ(X,N)
C
C        SUBROUTINE BESJ
C
C        PURPOSE
C           COMPUTE THE J BESSEL FUNCTION FOR A GIVEN ARGUMENT AND ORDER
C
C        USAGE
C           CALL BESJ(X,N,BJ,D,IER)
C
C        DESCRIPTION OF PARAMETERS
C           X  -THE ARGUMENT OF THE J BESSEL FUNCTION DESIRED
C           N  -THE ORDER OF THE J BESSEL FUNCTION DESIRED
C           BJ -THE RESULTANT J BESSEL FUNCTION
C           D  -REQUIRED ACCURACY
C           IER-RESULTANT ERROR CODE WHERE
C              IER=0  NO ERROR
C              IER=1  N IS NEGATIVE
C              IER=2  X IS NEGATIVE OR ZERO
C              IER=3  REQUIRED ACCURACY NOT OBTAINED
C              IER=4  RANGE OF N COMPARED TO X NOT CORRECT (SEE REMARKS)
C
C        REMARKS
C           N MUST BE GREATER THAN OR EQUAL TO ZERO, BUT IT MUST BE
C           LESS THAN
C              20+10*X-X** 2/3   FOR X LESS THAN OR EQUAL TO 15
C              90+X/2            FOR X GREATER THAN 15
C
C        SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED
C           NONE
C
C        METHOD
C           RECURRENCE RELATION TECHNIQUE DESCRIBED BY H. GOLDSTEIN AND
C           R.M. THALER,"RECURRENCE TECHNIQUES FOR THE CALCULATION OF
C           BESSEL FUNCTIONS",M.T.A.C.,V.13,PP.102-108 AND I.A. STEGUN
C           AND M. ABRAMOWITZ,"GENERATION OF BESSEL FUNCTIONS ON HIGH
C           SPEED COMPUTERS",M.T.A.C.,V.11,1957,PP.255-257
C
C
C
      COMMON /ERR/NER1,NER2,NER3,NER4
     2       /BJCOM/ BJDEL
      BJ=.0
       D = BJDEL
   20 IF(X) 25,30,31
   25 WRITE (3,900) X
   30 IF (N.EQ.0)  BJ = 1.0
      IF (N.GT.1)  GO TO 500
      RETURN
   31 IF(X-15.)32,32,34
   32 NTEST=20.+10.*X-X** 2/3
      GO TO 36
   34 NTEST=90.+X/2.
   36 IF(N-NTEST)45,500,500
   45 CONTINUE
      IER=0
      N1=N+1
      BPREV=.0
C
C     COMPUTE STARTING VALUE OF M
C
      IF(X-5.)50,60,60
   50 MA=X+6.
      GO TO 70
   60 MA=1.4*X+60./X
   70 MB=N+IFIX(X)/4+2
      MZERO=MAX0(MA,MB)
C
C     SET UPPER LIMIT OF M
C
```

```
      MMAX=NTEST
  100 DO 190 M=MZERO,MMAX,3
C
C     SET F(M),F(M-1)
C
      FM1=1.0E-28
      FM=.0
      ALPHA=.0
      IF(M-(M/2)*2)120,110,120
  110 JT=-1
      GO TO 130
  120 JT=1
  130 M2=M-2
      DO 160 K=1,M2
      MK=M-K
      BMK=2.*FLOAT(MK)*FM1/X-FM
      FM=FM1
      FM1=BMK
      IF(MK-N-1)150,140,150
  140 BJ=BMK
  150 JT=-JT
      S=1+JT
  160 ALPHA=ALPHA+BMK*S
      BMK=2.*FM1/X-FM
      IF(N)180,170,180
  170 BJ=BMK
  180 ALPHA=ALPHA+BMK
      BJ=BJ/ALPHA
      IF(ABS(BJ-BPREV)-ABS(D*BJ))200,200,190
  190 BPREV=BJ
      IER=3
      NER3 = NER3 + 1.
  200 RETURN
  500 WRITE(3,910) N
      STOP
  900 FORMAT(1H , 37HBESSEL FUNCTION ARGUMENT IS NEGATIVE , F12.3)
```

```
  910 FORMAT(1H , 30HBESSEL FUNCTION INVALID ORDER , I3)
      END
```

SUB-ROUTINE TIMER

```
      SUBROUTINE TIMER
   10 FORMAT ("4END OF JOB. TIME CONSUMED WAS"/"0 PROCESS..."I3" MINUTES
     1,"F6.2" SECONDS"/"0 IO........"I3" MINUTES,"F6.2" SECONDS")
      TIME2=TIME(2)/60.
      TIME3=TIME(3)/60.
      MTIME2=TIME2/60
      TIME2=AMOD(TIME2,60.)
      MTIME3=TIME3/60
      TIME3=AMOD(TIME3,60.)
      WRITE (3,10) MTIME2,TIME2,MTIME3,TIME3
      STOP
      RETURN
      END
```

APPENDIX B

DETERMINING MATERIAL PROPERTIES

B-1 Poro-elastic Material Constants

Seven material constants appear in the constitutive equations that were introduced in Chapter III. The seven constants are:

1) R , the pore compressibility
2) f , the porosity
3) a_2 , the shear modulus
4) a_3 , the apparent Lamé constant at steady state
5) $(a_4 - {}^{(2)}\bar{p} a_8)$, apparent Lamé constant at time zero for total stress tensor
6) a , the diffusion constant
7) a_1 , the fluid pressure interaction coefficient.

Certain material tests suggested by Biot and Willis [7] may be used to determine these constants. One of these is the jacketed compressibility test which consists of subjecting a sample of the porous fluid-filled material enclosed in a thin impermeable jacket to an external pressure p' . The inside of the jacket is vented to the atmosphere so that the excess pore pressure within the sample becomes negligible after a period of time. The specimen compresses, and water flows out of the pores during the test and eventually, a steady state condition is reached when the fluid flow ceases. The total sample volume change and total pore water outflow that occurs in reaching the steady state condition is measured. Using this information it is possible to compute k the coefficient of jacketed compressibility which is defined by Eq. (1).

$$k = -\frac{e_{mm}}{p'} \tag{1}$$

Substituting this expression in the total stress tensor constitutive equations, Eq. (6) of 4.1, and noting that π_{ij} is zero in this experiment when the steady state is reached, yields Eq. (2):

$$\frac{1}{k} = \frac{2}{3}a_2 + a_3 \tag{2}$$

which may be used to determine a_3 if the shear modulus a_2 is known.

Knowledge of the amount of water forced out of the sample during the jacketed compressibility test may be used to determine a_1. First, it is observed [7] that the measured water outflow per unit sample volume ζ is given by Eq. (3).

$$\zeta = f\,(e_{mm} - v_{m,m}) \tag{3}$$

Dividing Eq. (3) by e_{mm}, the measured change in sample volume per unit volume, noting that $\pi_{ij} = 0$ when steady state is reached, and using the last Eq. (34) of 3.4, yields Eq. (4).

$$\frac{\zeta}{e_{mm}} = \frac{f}{e_{mm}}\,(e_{mm} - v_{m,m}) = (1 + a_1)\,f \tag{4}$$

Since both ζ and e_{mm} are measured in the jacketed compressibility test, it follows that if the porosity f is known the constant a_1 can be obtained using Eq. (4).

A second test, the unjacketed compressibility test is required, if the skeletal material is compressible, to determine the apparent Lame constant at time zero for the total stress tensor. This test consists of applying a hydrostatic stress p' to a saturated sample of the material under undrained conditions. The sample volume change that results is measured. An unjacketed compressibility constant is defined as shown in Eq. (5).

$$\delta = -\frac{e_{mm}}{p'} \tag{5}$$

Using Eqs. (34) of 3.4 leads to Eq. (6):

$$\frac{1}{\delta} = \frac{2}{3}a_2 + a_4 - {}^{(2)}\bar{\rho}\,a_8 \tag{6}$$

which yields the desired Lamé constant if the shear modulus a_2 is known.

The pore compressibility constant R may be computed by combining some of the information obtained in the jacketed test with that obtained in the unjacketed test. First it is observed that by definition R may be expressed as shown in Eq. (7).

$$R = \frac{\zeta}{(e_{mm} - e_{mm}|_{t=0})} \tag{7}$$

The instantaneous dilatation in the jacketed test may be computed using unjacketed compressibility constant as shown in Dq. (8).

$$e_{mm}\big|_{t=0} = -p'\delta \tag{8}$$

Since the value of e_{mm} is measured in the jacketed test, R may be computed.

The porosity constant, f may be obtained by first determining the specific gravity $S.G.$ of the skeletal material contained in a sample of the poro-elastic mixture. Then the porosity, or void space per unit volume of the skeletal material can be computed from the specific gravity information as shown in Eq. (9).

$$f = 1 - \frac{(\textit{Sample Weight (dry) / Unit Vol.})}{\textit{Density of Water}} / S.G. \tag{9}$$

The specific gravity of soil materials which are of primary interest in this study should be determined in accordance with ASTM Specification D 854-58 (AASHO No. T 100).

The diffusion constant a is related to the Darcy coefficient of permeability $D.C.P.$ by Eq. (10):

$$a = \frac{\textit{fluid density}}{(D.C.P.)\, f} \tag{10}$$

where the fluid density is that of the fluid contained in the pores. The permeability constant for granular soils should be determined according to ASTM D 2434-68.

Finally, it is necessary to determine the shear modulus a_2 . One way of determining this constant is to superimpose a unixial stress in addition to the hydrostatic pressure p' , at the end of the jacketed compressibility test. Assuming the additional loading is applied in the z direction, a_2 may be determined from Eq. (11):

$$a_2 = \frac{(\Delta\sigma_{zz} - a_3 \Delta e_{mm})}{2\Delta e_{zz}} \tag{11}$$

which follows from Eq. (6) of 4.1. The axial strain Δe_{zz} in Eq. (11) may be determined by measuring the axial deformation of the sample.

B-2 Stress Relaxation of Sand-Asphalt Mixtures

The results of stress relaxation measurements on a sand-asphalt mixture that were obtained by Moavenzadeh and Soussou [19] are used in the example problem that is presented in Chapter X of this report. These particular data were chosen because of the convenient mathematical form in which they were presented. It would be more physically realistic to utilize data from tests on bituminous concrete because flexible pavements are constructed primarily of such bituminous and coarse aggregate mixtures. With regard to this it is noted that the quasi-static response of sand-asphalt mixtures and of bituminous concrete has been studied by several other investigators in recent years [20, 21, 22]. The data from tests on these other mixtures could be put in the same form as that given in reference [19] by using a numerical collocation technique that will be described here.

The sand-asphalt mixture used in the Moavenzadeh study was made up of Ottawa sand and asphalt cement. The gradation of the Ottawa sand used is given in Table 1.

TABLE 1
GRADATION OF AGGREGATES

Sieve Size	Percent Passing*	
	ASTM D 1663-59T	Selected Gradation
16	85-100	100
30	70- 95	75
50	45- 75	45
100	20- 40	26
200	9- 20	15

* Percent of the total weight of material passing a given sieve size.

The asphalt cement used was an AC-20 grade asphalt and the results of some conventional tests on the material are given in Table 2.

TABLE 2
RESULTS OF TESTS ON ASPHALT

Test	Result
Specific gravity, 77/77 F	1.020
Penetration, 200 gm, 60 sec, 39.9	30
Ductility, 77 F	250 + CM
Flash point, Cleveland open cup	545 F

Relaxation test data were obtained for the mixture by subjecting it to uniaxial compressive strain. Corrections were introduced to account for the fact that the initial loading near time zero was not constant but was increased from zero to the desired constant level linearly with time. An exponential series function of the form Eq. (12) :

$$\Phi(t) = \Phi(\infty) + \sum_{i=1}^{j} B_i e^{-b_i t} \quad (12)$$

was fitted to the stress response versus time data using a technique developed by Schapery [23]. Basically the technique consists of selecting the b_i coefficients to span several decades of time, and then selecting the B_i to minimize the mean square error between the function and the data.

A ten term series was selected to model the sand asphalt mixture, with the b_i coefficients defined by (13).

$$b_i = 5 \times 10^4 \times 10^{-i}, \quad i = 1 \text{ to } 10 \quad (13)$$

Fitting the curve to the data by the least squares technique yielded the B_i coefficients as indicated in Table 3.

TABLE 3
NUMERICAL VALUES OF COEFFICIENTS
OF STRESS RELAXATION FUNCTION

$$\phi(t) = \phi(\infty) + \sum_{i=1}^{10} B_i e^{-b_i t}$$

where:	B_1	$= 6.65 \times 10^3$	b_1	$= 5 \times 10^3$
	B_2	$= 2.32 \times 10^4$	b_2	$= 5 \times 10^2$
	B_3	$= 4.19 \times 10^4$	b_3	$= 5 \times 10^1$
	B_4	$= 3.30 \times 10^4$	b_4	$= 5 \times 10^0$
	B_5	$= 3.01 \times 10^4$	b_5	$= 5 \times 10^{-1}$
	B_6	$= 1.26 \times 10^4$	b_6	$= 5 \times 10^{-2}$
	B_7	$= 4.30 \times 10^3$	b_7	$= 5 \times 10^{-3}$
	B_8	$= 1.44 \times 10^3$	b_8	$= 5 \times 10^{-4}$
	B_9	$= 8.36 \times 10^2$	b_9	$= 5 \times 10^{-5}$
	B_{10}	$= 4.03 \times 10^2$	b_{10}	$= 5 \times 10^{-6}$
	B_∞	$= 1.60 \times 10^3$		

APPENDIX C

INITIAL AND STEADY STATE SOLUTIONS

The solutions corresponding to the Laplace-Hankel transformed solution images can be readily determined for the initial state and steady state [24] by computing the limits shown in Eq. (1).

$$g_\nu(\eta,t)\Big|_{t=0} = \lim_{s\to\infty} s\,\bar{g}_\nu(\eta,s)\ , \qquad \textit{initial value case}$$

$$g_\nu(\eta,t)\Big|_{t=\infty} = \lim_{s\to 0} s\,\bar{g}_\nu(\eta,s)\ , \qquad \textit{steady state case} \tag{1}$$

The initial value and steady state transforms can then be inverted numerically by approximating the inverse Hankel transformation operation shown in Eq. (2).

$$f(r,t)\Big|_{t=\infty} = \lim_{\sigma\to\infty} \int_0^{\sigma} g_\nu(\eta,0)\,\eta J_\nu(\eta,r)\,d\eta \tag{2}$$

The limits indicated in Eq. (1) have been analytically determined for the double transforms that were obtained in Chapter VIII. The limit functions obtained are presented here.

The initial value limiting functions can be predicted independently because the initial values of all the dependent variables can be obtained in the same manner that the initial value of the dilatation was determined in 7.2. The initial values of the Hankel transforms of the dependent variables obtained using limiting process Eq. (1) are given in Eq. (3).

$$\lim_{s \to \infty} s\bar{q}_0^*(\eta,s) = -kc(a_2 + \alpha_4 - \bar{\rho}^{(2)}\alpha_8) \tag{a}$$

$$\lim_{s \to \infty} s\bar{w}_0(\eta,s) = \frac{kc}{2a_2\eta}(2a_2 + \alpha_4 - \bar{\rho}^{(2)}\alpha_8) \tag{b}$$

$$\lim_{s \to \infty} s\,\bar{u}_{z_0}(\eta,z,s) = \left[\frac{kc(2a_2 + \alpha_4 - \bar{\rho}^{(2)}\alpha_8)(1+\eta z)}{2a_2\eta} - \frac{kcz}{2}\right]e^{-\eta z} \tag{c}$$

$$\lim_{s \to \infty} s\bar{e}_{mm_0}(\eta,z,s) = -kce^{-\eta z} \tag{d}$$

$$\lim_{s \to \infty} s\,\bar{P}_0(\eta,z,s) = \frac{kce^{-\eta z}}{(1+a_1)}(-a_3 + \alpha_4 - \bar{\rho}^{(2)}\alpha_8) \tag{e}$$

$$\lim_{s \to \infty} s\,\bar{\sigma}_{mm_0}(\eta,z,s) = \frac{-kce^{-\eta z}}{(1+a_1)}\Big[a_1(2a_2 + 3\alpha_4 - 3\bar{\rho}^{(2)}\alpha_8) + (2a_2 + 3a_3)\Big] \tag{f}$$

$$\lim_{s \to \infty} s(\bar{\sigma}_{mm_0}(\eta,z,s) - 3\bar{P}_0(\eta,z,s)) = -kce^{-\eta z}(2a_2 + 3(\alpha_4 - \bar{\rho}^{(2)}\alpha_8)) \tag{g}$$

(3)

$$\lim_{s \to \infty} s\bar{\sigma}_{zz_0}(\eta,z,s) = -kce^{-\eta z}\Big[\eta z(a_2 + \alpha_4 - \bar{\rho}^{(2)}\alpha_8) + \frac{(a_3 + a_1(\alpha_4 - \bar{\rho}^{(2)}\alpha_8))}{(1+a_1)} + a_2\Big] \tag{h}$$

$$\lim_{s \to \infty} s(\bar{\sigma}_{zz_0}(\eta,z,s) - \bar{P}_0(\eta,z,s)) = -kce^{-\eta z}(a_2 + \alpha_4 - \bar{\rho}^{(2)}\alpha_8)(\eta z + 1) \tag{i}$$

$$\lim_{s \to \infty} \bar{u}_{r_1}(\eta,z,s) = \frac{e^{-\eta z}}{2a_2\eta}\Big[kc(a_2 + \alpha_4 - \bar{\rho}^{(2)}\alpha_8)\ \eta z - kca_2\Big] \tag{j}$$

$$\lim_{s \to \infty} \bar{\sigma}_{rz_1}(\eta,z,s) = -\eta kc(a_2 + \alpha_4 - \bar{\rho}^{(2)}\alpha_8)e^{-\eta z}z \tag{k}$$

$$\lim_{s \to \infty} \frac{sd\bar{\sigma}_{rz_1}(\eta,z,s)}{dz} = -\eta e^{-\eta z}kc(a_2 + \alpha_4 - \bar{\rho}^{(2)}\alpha_8)(1-\eta z) \tag{l}$$

$$\lim_{s \to \infty} s(\bar{\sigma}_{rr_0} - 2\bar{P}_0 + \sigma_{\theta\theta_0}) = -kce^{-\eta z}\Big[a_2 + 2(\alpha_4 - \bar{\rho}^{(2)}\alpha_8) + \eta z(a_2 + \alpha_4 - \bar{\rho}^{(2)}\alpha_8)\Big] \tag{m}$$

Using Terezawa's solution [12] the initial condition defining problem of 7.2 was solved and expressions Eq. (3a), (b), (d), (g), (i), (j), and (k) were obtained by this alternative procedure. These results verify the validity of the computed limits indicated.[1] Expression Eq. (3e), (f), and (h), are either used in combination to make up the other expressions or are made up of a combination of the others. Therefore, the results obtained by direct solution imply that Eq. (3e), (f), and (h) are valid also. Examining Eq. (3a), (b), (c), (e), (h), and (k) it can be seen that the boundary conditions at $z = 0$ are satisfied at $t = 0$ as summarized in Eq. (4a), (b), and (c).

$$\lim_{s\to\infty} s\bar{q}_0^+(\eta,s) = \lim_{s\to\infty} s(\bar{\sigma}_{zz_0}(\eta,s) - \bar{P}_0(\eta,s))\Big|_{z=0} \quad \text{(a)}$$

$$\lim_{s\to\infty} s\bar{w}_0(\eta,s) = \lim_{s\to\infty} s\bar{u}_{z_0}(\eta,s)\Big|_{z=0} \quad \text{(b)} \qquad (4)$$

$$\lim_{s\to\infty} s\,\bar{\sigma}_{rz_1}(\eta,s)\Big|_{z=0} = 0 \quad \text{(c)}$$

The steady state Hankel transforms of the dependent variables are given by Eqs. (5) and (6).

$$K \equiv \frac{q\,b\,J_1(b\eta)\,2a_2}{\left[DR(\infty)\eta^4(2a_2+a_3) + 2a_2\eta(a_2+a_3)\right]c} \qquad (5)$$

$$\lim_{s\to 0} s\,\bar{q}_0^+(\eta,s) = -Kc(a_2+a_3) \quad \text{(a)}$$

$$\lim_{s\to 0} s\,\bar{w}_0^+(\eta,s) = \frac{(2a_2+a_3)}{2a_2\eta} Kc \quad \text{(b)} \qquad (6)$$

$$\lim_{s\to 0} s\,\bar{u}_{z_0}(\eta,z,s) = e^{-\eta z}\left[\frac{Kc(2a_2+a_3)(1+\eta z)}{2a_2\eta} - \frac{kcz}{2}\right] \quad \text{(c)}$$

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[1] The plate deflection and foundation reaction at time zero, given by Eqs. (3a) and (3b), also agrees with the known solution for an elastic plate supported on an elastic half-space and subjected to a circular uniform load [25].

$$\lim_{s \to 0} s\,\bar{e}_{mm_0}(\eta,z,s) = -Kce^{-\eta z} \qquad \text{(d)}$$

$$\lim_{s \to 0} s\,\bar{P}_0(\eta,z,s) = 0 \qquad \text{(e)}$$

$$\lim_{s \to 0} s\,\bar{\sigma}_{mm_0}(\eta,z,s) = -(2a_2 + 3a_3)Kce^{-\eta z} \qquad \text{(f)}$$

$$\lim_{s \to 0} s\,\bar{\sigma}_{zz_0}(\eta,z,s) = ce^{-\eta z}\left[ka_2\eta z - K(a_2 + a_3) - K(2a_2 + a_3)\eta z\right] \qquad \text{(g)}$$

$$\lim_{s \to 0} s\,\bar{u}_{r_1}(\eta,z,s) = \frac{e^{-\eta z}}{2a_2\eta}\left[K(2a_2 + a_3)\eta z - Kca_2 - kca_2\eta z\right] \qquad \text{(h)} \qquad (6)$$

$$\lim_{s \to 0} s\,\bar{\sigma}_{rz_1}(\eta,z,s) = kca_2e^{-\eta z}\eta z - Kc(2a_2 + a_3)e^{-\eta z}\eta z \qquad \text{(i)}$$

$$\lim_{s \to 0} s\,\frac{d\bar{\sigma}_{rz_1}(\eta,z,s)}{dz} = \eta e^{-\eta z}\left[Kc(2a_2 + a_3)(1-\eta z) - Kca_2 + kca_2\eta z\right] \qquad \text{(j)}$$

$$\lim_{s \to 0} s\,(\bar{\sigma}_{rr_0} - 2\bar{P}_0 + \bar{\sigma}_{\theta\theta_0}) = -Kc(a_2 + 2a_3)e^{-\eta z} + Kc(2a_2 + a_3)\,\eta z e^{-\eta z} - kca_2\eta z e^{-\eta z} \qquad \text{(k)}$$

Examining Eq. (6a), (e), and (g) verifies that boundary condition Eq. (7a) is satisfied. Using Eq. (6b) and (c) with z set at zero, it can be seen that Eq. (7b) is satisfied also. Setting z equal to zero in Eq. (6i) shows that boundary condition Eq. (7c) is satisfied in the steady state.

$$\lim_{s \to 0} s\,\bar{q}_0^{+}(\eta,s) = \lim_{s \to 0} s\,(\bar{\sigma}_{zz_0}(\eta,z,s) - \bar{P}_0(\eta,z,s))\Big|_{z=0} \qquad \text{(a)}$$

$$\lim_{s \to 0} s\,\bar{w}_0^{+}(\eta,s) = \lim_{s \to 0} s\,\bar{u}_{z_0}(\eta,z,s)\Big|_{z=0} \qquad \text{(b)} \qquad (7)$$

$$\lim_{s \to 0} s\,\bar{\sigma}_{rz_1}(\eta,z,s)\Big|_{z=0} = 0 \qquad \text{(c)}$$

In this appendix Hankel transforms of the steady state and initial values of the solutions to the plate on foundation problems have been presented. The initial value transforms have been compared with independently determined results and have been found to be correct. The transformed steady state solutions are utilized in the computer program.